The Capitalist Spirit: Optimal Consumption and Investment When Wealth Enters Utility Directly *

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April 23, 2013

Abstract

This paper studies a continuous-time model of consumption and investment with a utility function where wealth enters as an argument. The stochastic maximum principle is used to obtain a solution and compared with a newly formulated martingale method. A concrete solution is obtained by a discrete time approximation. An agent with such a utility function typically exhibits decreasing marginal propensity to consume and an increasing marginal propensity to invest in the risky assets consistent with empirical evidence.

1 Introduction

A century ago Max Weber argued that the spirit of capitalism had lied in striving for accumulating wealth for its own sake, not for pleasure of consumption which the wealth could bring about [1, 10, 2]. Recently, Carroll adopted the idea and proposed a model in which wealth enters consumers' utility functions directly [5]. He used the utility functions to explain the seemingly puzzling behaviour of *the rich*: their savings had been large [8, 5] and their investments in the risky assets had been also large despite the fact that most of them had embraced un-diversifiable risks in the form of ownership of proprietorship [6].

*The work is supported by Basic Research Program of China (973 Program, No.2007CB814904), the Science Foundation for Ministry of Education of China (No.20090071110001), WCU (World Class University) Program through the Korea Science and Engineering Foundation funded by the Ministry of Education, Science and Technology (R31-20007), NNSF of China (No.10901060, No.10971073), NNSF of Guang-Dong province (No.9451063101002091), Shanghai Postdoctoral Science Foundation (No.10R21411100) and China Postdoctoral Science Foundation Funded Project (No.201003244, 20100480555), Guangdong InnovativeUniversities in Outstanding Young Talent Cultivation Project (No. LYM10062).

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Note that there are several models in the literatures about the explanation for wealth accumulation: the *Life Cycle* model, the *Dynastic* model, and the *Joy of Giving* model. Roughly speaking, the Life Cycle model says that people accumulates the wealth to finance the future consumption. Even though the pursuit of consumption may be a fundamental need for human being, assuming that is for *necessary* goods, this was empirically shown not to be a main reason for some people who has large wealth, *the rich*. A possible alternative is the Dynastic model, which says the wealth is pursued for the inheritance to his descendants. However, this model is also poor with empirical evidence. On the other hand, the Joy of Giving model directly introduces a utility from wealth. Carroll[5] presents a simple two period example to reason the Joy of Giving model.

The purpose of this paper is to investigate an optimal consumption and investment strategies of an economic agent with "the capitalist spirit," that is, with a utility function as proposed by Carroll. Such an agent typically exhibits *decreasing marginal propensity to consume(MPC)* and *increasing marginal propensity to invest in the risky assets(MPIR)* consistent with empirical evidence. Moreover, the maximum principle, different from the dynamic programming method ordinarily employed in the study of consumption and portfolio selection problems, has good prospects for applicability to similar problems.

Max Weber's explanation of the capitalist spirit was religions, claiming that the ascetic nature of protestantism was the driving force of the capitalism: a religious individual abstained from earthly pleasures but pursued wealth in an effort to ascertain his/her salvation. A secular and perhaps more reasonable explanation can be provided by a motive derived from "social interaction" as proposed by Becker [3]: recognition from others in social interaction was an important objective of human pursuit. Wealth is an important factor in social recognition and thus can enter into human utility functions. Of course, one can argue that social recognition is derived from gifts and other altruistic activities for which wealth can be used as a tool not from wealth *per se* and thus a utility function where wealth enters directly is only a reduced form derived from a fundamental utility function which has gifts, beneficent activities, and etc. as arguments. One cannot, however, deny that social recognition is attached to large wealth itself independent of good deeds of the person who owns it. Thus, a utility function with wealth as an argument is one natural representation of a human preference which strives for social recognition in a society where such recognition is automatically given to a person with large wealth.

So the agent's problem is to choose an optimal portfolio rule which maximizes his utility of terminal wealth, that of consumption, and that of wealth such as

$$\max_{\pi,C} E\left[e^{-\beta T}M\left(W_T\right)+\int_0^T e^{-\beta s}U\left(C_s,W_s\right)ds\right]. \tag{1.1}$$

The wealth utility can be directly interpreted as the happiness of the wealth itself which stems from the *social recognition* in the *social rank*. The difference between the Carroll's model and the one here is that we consider the *running* utility of wealth during the lifetime, which makes an economically intriguing problem.

We approach this problem by *Stochastic Maximum Principle(SMP)* and a *Martingale Method*. As we shall see in the following section, the SMP gives a *coupled Forward-Backward Stochastic Differential Equation(FBSDE)* to get the optimal solutions, which consists of the wealth dynamics and its adjoint process. We also proposes a Martingale method which is an alternative approach to solve the problem by using the *Lagrangian* method.

For the numerical example, we examine multi-period models by using the binomial tree model. We solve the problem numerically for both finite horizon and infinite horizon by discrete time approximation. We use the *CRRA(Constant Relative Risk Aversion)* utility function for consumption and the *HARA(Hyperbolic Absolute Risk Aversion)* utility function for wealth. We compare the optimal solutions with Merton problem as a benchmark.

In section two, we formulate the problem by using SMP and derive the optimality conditions. In section three, we propose a Martingale method and shows that the two methods are equivalent. Plus, we reason the economic meaning for a new budget constraint. We also approach the problem by the binomial tree model and show the optimality conditions in discrete time. In section four, we present multi-period numerical examples for both finite horizon problem and infinite horizon problem. The final section is the conclusion.

2 Stochastic Maximum Principle

A robust framework to solve an optimal control problem is Stochastic Maximum Principle(SMP). In this section, we formulate the problem by using a SMP.

The financial market consists of one riskless asset and N risky assets. The price $S_0(t)$ of the riskless asset evolves as follows:

$$\frac{dS_0(t)}{S_0(t)} = r(t)dt, \qquad S_0(0) = s_0 \tag{2.2}$$

where $r(t)$ is the instantaneous riskfree rate. The vector $S(t) = (S_1(t), \ldots, S_N(t))'$ of prices of N risky assets evolves as

$$\frac{dS(t)}{S(t)} = \mu(t)dt + \sigma(t)dB(t), \qquad S(0) = s_0 \tag{2.3}$$

where $\mu(t)$ is an N-vector, $\sigma(t)$ is an $N \times M$ matrix, $B(t)$ is an N-dimensional standard Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, P)$. We assume filtration $\mathcal{F}$ is generated by $B(t)$ and augmented by null sets. In (2.3), a symbol $'$ denotes the transpose of a matrix and the fraction of vectors means componentwise operation, that is, $\frac{dS(t)}{S(t)} = \left(\frac{dS_1(t)}{S_1(t)}, \ldots, \frac{dS_N(t)}{S_N(t)}\right)'$. We assume that processes r, μ, and σ are adapted to filtration $\mathcal{F}$. We will assume that the σ is invertible, that is, the financial market is complete.

Now we apply a stochastic maximum principle. The maximum principal was originally developed by Pontriagyn and Boltyanskii for the deterministic case. Here we use Bismut's generalization of the principle to a stochastic case [4].

Let us introduce an economic agent who lives for a period $[0, T]$. The *cost functional* is stated as follows.

$$J(\pi, C) = E\left[e^{-\beta T} M(W_T) + \int_0^T e^{-\beta t} U(C_t, W_t)\, dt\right] \tag{2.4}$$

where $\beta > 0$ is the agent's subjective discount rate. The system dynamics of wealth is

$$\begin{cases} dW_t &= [r_t W_t + \pi_t(\mu_t - r_t) - C_t]\, dt + \pi_t \sigma_t dB_t \\ W_0 &= w_0. \end{cases} \tag{2.5}$$

Then the *adjoint process* of the dynamics is

$$\begin{cases} dp_t &= -\left[r_t p_t + e^{-\beta t} U_W(\overline{C}_t, \overline{W}_t)\right] dt - \theta_t p_t dB_t \\ p_T &= e^{-\beta T} M_W(\overline{W}_T), \end{cases} \tag{2.6}$$

which can be expressed in an integral from such as

$$p_t = e^{-\beta T} M_W(\overline{W}_T) + \int_t^T \left[r_t p_t + e^{-\beta(s-t)} U_W(\overline{C}_s, \overline{W}_s)\right] ds - \int_t^T q_s dB_s$$

where

$$\theta_t \triangleq \sigma_t^{-1}(\mu_t - r\mathbf{1})$$

and $q_s \triangleq -\theta_t p_t$ for $t \in [0, T]$. Here $\mathbf{1}$ is the N-vector of 1's, and θ_t is the market price of risk.

The *Hamiltonian* is

$$H(t, W, \pi, C; p, q) \triangleq p(r_t W + (\mu_t - r_t)\pi - C) + q\sigma_t \pi + e^{-\beta t} U(C, W). \tag{2.7}$$

Then the *maximum conditions* are the following.

$$\max \left\langle \begin{pmatrix} H_\pi \\ H_C \end{pmatrix} (t, \overline{W}_t, \overline{\pi}_t, \overline{C}_t; p_t, q_t),\ \begin{pmatrix} \pi - \overline{\pi}_t \\ C - \overline{C}_t \end{pmatrix} \right\rangle_{\mathbb{R}^2} \geq 0, \tag{2.8}$$

which imply

$$\begin{aligned} H_\pi(t, \overline{W}_t, \overline{\pi}_t, \overline{C}_t; p_t, q_t) &= p_t(\mu_t - r_t) + q_t \sigma_t = 0 \\ H_C(t, \overline{W}_t, \overline{\pi}_t, \overline{C}_t; p_t, q_t) &= -p_t + e^{-\beta t} U_C(\overline{C}_t, \overline{W}_t) = 0. \end{aligned}$$

Therefore,

$$e^{-\beta t} U_C(\overline{C}_t, \overline{W}_t) = p_t.$$

Hence we have to solve a *coupled Forward-Backward Stochastic Differential Equation(FBSDE)* which consists of the equation (2.5) and (2.6) to get the optimal solution. Note that the Merton problem gives a *decoupled FBSDE* which can be solved independently with each other. On the other hand, solving the *coupled FBSDE* can be an independent work beyond the scope of this study. However, we can try another approach which is fundamentally equivalent with the SMP to solve the problem. If we find a solution form for the adjoint equation, we get

$$\begin{aligned} p(t) &= p(0) e^{-\int_0^t \left[r(s) + \frac{1}{2}\theta(s)^2\right] ds + \int_0^t \theta(s) dB(s)} \\ &\quad - \int_0^t U_W(\overline{C}_s, \overline{W}_s) e^{-\beta s} e^{-\int_s^t \left[r(\tau) + \frac{1}{2}\theta(\tau)^2\right] d\tau + \int_0^s \theta(\tau) dB(\tau)} ds. \end{aligned}$$

Let $y(t) \triangleq e^{\beta t}p(t)$. Then,

$$\begin{aligned} d(y(t)) &= \beta e^{\beta t}p(t)dt + e^{\beta t}dp(t) \\ &= \beta e^{\beta t}p(t)dt - e^{\beta t}(r(t)p(t) + e^{-\beta t}U_W)dt - e^{\beta t}\theta(t)p(t)dB(t) \\ &= (\beta - r(t))e^{\beta t}p(t)dt - U_W dt - e^{\beta t}p(t)\theta(t)dB(t) \\ &= (y(t)(\beta - r(t)) - U_W)\, dt - y(t)\theta(t)dB(t) \end{aligned}$$

$$\therefore y(t) = y(0)e^{\beta t}\xi(t) - \int_0^t U_W \frac{\xi(t)}{\xi(u)} du \tag{2.9}$$

where

$$\xi(t) \triangleq e^{-\int_0^t \left[r(\tau) + \frac{1}{2}\theta(\tau)^2\right]d\tau + \int_0^t \theta(\tau)dB(\tau)}.$$

Thus

$$U_c(\overline{C}_t, \overline{W}_t) = y_t. \tag{2.10}$$

We will show that this makes the SMP approach for this problem equivalent to a Martingale approach so that we can solve the problem by using the Lagrangian method.

3 A Martingale Approach

In this section, we propose a Martingale method to solve the problem, and then we get the optimality conditions for multi-period problems by discrete time approximation.

We now provide a modification of the martingale approach which has the same final result as the maximum condition explained in the previous section. The necessary modification is to consider a constraint for each wealth chosen. Since wealth is a choice variable for each time and state, the number of constraints is uncountable. We introduce it in an integral form in the following Lagrangian:

$$\begin{aligned} \mathcal{L} &= E\left\{\int_0^\infty e^{-\beta t}U(C(t), W(t))dt + \lambda\left[W(0) - \int_0^\infty \xi(t)C(t)dt\right]\right. \\ &\qquad \left. - \int_0^\infty \eta(t)\left[\xi(t)W(t) - \int_t^\infty \xi(s)C(s)ds\right]dt\right\} \end{aligned} \tag{3.11}$$

in case of no bequest utility. Note that we put a minus sign before $\eta(t)$, the Lagrange multiplier of the last constraint in the Lagrangian. The reason for this choice is to make the multiplier positive as explained below. Then the optimality conditions are the following.

$$\begin{aligned} U_C(C(t), W(t)) &= \lambda\xi(t)e^{\beta t} - \int_0^t \eta(u)\xi(t)e^{\beta t}du \\ &= \xi(t)e^{\beta t}\left[\lambda - \int_0^t \eta(u)du\right] \\ U_W(C(t), W(t)) &= \eta(t)e^{\beta t}\xi(t) \end{aligned}$$

$$\therefore \eta(t) = e^{-\beta t}\frac{1}{\xi(t)}U_W(C(t), W(t)).$$

Note that $U_W(C(t), W(t)) > 0$. Thus, the minus sign before $\eta(t)$ is justified.

$$U_C(C(t), W(t)) = \xi(t)e^{\beta t}\left[\lambda - \int_0^t \frac{e^{-\beta u}}{\xi(u)}U_W(C(u), W(u))du\right]. \tag{3.12}$$

Note that the equation (2.10) and (3.12) shows the same result. Hence the SMP and the Martingale approach are equivalent here.

Note that we have an additional constraint for this problem such that

$$E^t \int_t^\infty \xi(s)C(s)ds \geq \xi(t)W(t). \tag{3.13}$$

The equation (3.13) is different with the traditional budget constraint such that the initial wealth is bounded below by the value of the future consumption bundle. The new constraint says that the value of the future consumption bundle has to be bounded below by some current wealth level at *each time*. This constraint naturally occurs because now the agent needs to consider the utility of consumption and that of wealth at each time through his lifetime. We dig the explanation more with a two period model.

3.1 A Two Period Model

Here we consider a two-period model of the problem stated previously. It is a binomial model popular in option pricing. There are two time periods, 0 and 1, and two assets, a riskless asset and a risky asset. One dollar invested in the riskless asset at time 0 gives a return equal to $R = 1 + r$ at time 1 for $r > 0$. The same one dollar invested in the risky asset at time 0 generates a return equal either to u or d at time 1, where $u > R > d > 0$. Thus there exist two states of the world according to the return of the risky asset at time 1. By a slight abuse of notation we will denote the state in which the risky asset's return is $u(d)$ by $u(d)$.

For simplicity of exposition we assume the following separable form of the felicity function:

$$\tilde{U}(C_t, W_t) = U(C_t) + V(W_t). \tag{3.14}$$

In a two period model, we have

$$W_0\Delta t = C_0\Delta t + W_0^a\Delta t \left\langle \begin{array}{l} W_u\Delta t = C_u\Delta t + W_u^a\Delta t \\ W_d\Delta t = C_d\Delta t + W_d^a\Delta t \end{array}\right.$$

where subscript $u(d)$ means consumption or wealth in state $u(d)$ and superscript a denotes wealth after consumption. For the discrete time model, we have to explicitly distinguish the given wealth, the wealth before consumption, and the optimal wealth, the wealth after consumption. The difference vanishes as the time goes to continuous time. Then the agent's problem is to

$$\max_{C_0, C_u, C_d, W_0^a, W_u^a, W_d^a} U(C_0)\Delta t + V(W_0^a)\Delta t + \beta E\left[U(C_1) + V(W_1^a)\right]\Delta t$$

subject to

$$\begin{aligned} W_0 \Delta t &= C_0 \Delta t + \pi_u \xi_u (C_u \Delta t + W_u^a \Delta t) + \pi_d \xi_d (C_d \Delta t + W_d^a \Delta t) \\ W_0^a \Delta t &= \pi_u \xi_u (C_u \Delta t + W_u^a \Delta t) + \pi_d \xi_d (C_d \Delta t + W_d^a \Delta t) \end{aligned}$$

where the *state price densities* are

$$\xi_u = \frac{p_u}{R\pi_u} \qquad \xi_d = \frac{p_d}{R\pi_d}$$

and the *risk-neutral* probabilities are

$$p_u = \frac{R-d}{u-d} \qquad p_d = \frac{u-R}{u-d}.$$

Define the *Lagrangian* as

$$\begin{aligned} \mathcal{L} &= U(C_0)\Delta t + V(W_0^a)\Delta t + \beta E\left[U(C_1) + V(W_1^a)\right]\Delta t \\ &\quad + \lambda \left\{ W_0 \Delta t - C_0 \Delta t - \pi_u \xi_u (C_u + W_u^a)\Delta t - \pi_d \xi_d (C_d + W_d^a)\Delta t \right\} \\ &\quad - \eta_0 \Delta t \left\{ W_0^a - \pi_u \xi_u (C_u + W_u^a)\Delta t - \pi_d \xi_d (C_d + W_d^a)\Delta t \right\}. \end{aligned}$$

The first order conditions(FOCs) are the following.

$$\begin{aligned} C_0 &: U_C(\overline{C}_0)\Delta t = \lambda \Delta t \\ C_u &: U_C(\overline{C}_u)\Delta t = \frac{\xi_u}{\beta}\left(\lambda - \eta_0 \Delta t\right)\Delta t \\ C_d &: U_C(\overline{C}_d)\Delta t = \frac{\xi_d}{\beta}\left(\lambda - \eta_0 \Delta t\right)\Delta t \\ W_0^a &: V_W(\overline{W}_0^a)\Delta t = \eta_0 \Delta t \\ W_u^a &: V_W(\overline{W}_u^a)\Delta t = \frac{\xi_u}{\beta}\left(\lambda - \eta_0 \Delta t\right)\Delta t \\ W_d^a &: V_W(\overline{W}_d^a)\Delta t = \frac{\xi_d}{\beta}\left(\lambda - \eta_0 \Delta t\right)\Delta t. \end{aligned}$$

Therefore,

$$\begin{aligned} U_C(\overline{C}_u) &= V_W(\overline{W}_u^a) \\ U_C(\overline{C}_d) &= V_W(\overline{W}_d^a). \end{aligned}$$

Note that $\eta_0 = V_W(\overline{W}_0^a) > 0$. Thus, the minus sign before η_0 is justified. The fact $\eta_0 > 0$ is puzzling since it implies the constraint in inequality should be written as

$$W_0^a \leq \pi_u \xi_u (C_u + W_u^a) + \pi_d \xi_d (C_d + W_d^a). \tag{3.15}$$

After a thought, however, the meaning is clear. When the agent chooses W_0^a, wealth after consumption which contributes to his utility because of the prestige or social recognition it generates, by sacrificing current consumption, he essentially commits to have more consumption and wealth in the future. Thus, the choice of wealth implies a lower bound for the present value of future consumption and wealth, not an upper bound. This is a special feature of the problem.

Note that there is an intertemporal linkage of choices. The period one consumption is dependent on the choice of period 0 not through period 1 wealth alone, but through its effect on its marginal utility. That is η_0 enters in the calculation of marginal utility of consumption at time 1. This makes the problem *non-Markovian* and makes an ordinary dynamic programming method inapplicable.

Let us fix the marginal utility of consumption at period 0, $\lambda > 0$, then the fundamental pricing equation in the binomial model gives

$$\begin{aligned} \frac{p_u}{R}\left[\overline{C}_u(\eta_0(\lambda)\Delta t)\Delta t + \overline{W}^a_u(\eta_0(\lambda)\Delta t)\Delta t\right] + \frac{p_d}{R}\left[\overline{C}_d(\eta_0(\lambda)\Delta t)\Delta t + \overline{W}^a_d(\eta_0(\lambda)\Delta t)\Delta t\right] &= \overline{W}^a_0\Delta t \\ \Rightarrow V_W\left\{\frac{p_u}{R}\left[\overline{C}_u(\eta_0(\lambda)\Delta t) + \overline{W}^a_u(\eta_0(\lambda)\Delta t)\right] + \frac{p_d}{R}\left[\overline{C}_d(\eta_0(\lambda)\Delta t) + \overline{W}^a_d(\eta_0(\lambda)\Delta t)\right]\right\} &= V_W(\overline{W}^a_0) \\ &= \eta_0 \quad (3.16) \end{aligned}$$

in the absence of the arbitrage opportunity. Assume for now that the existence of a unique solution to the equation and denote it by $\eta_0(\lambda)$. Then,

$$\begin{aligned} \overline{C}_u &= U_C^{-1}\left(\frac{\xi_u}{\beta}(\lambda - \eta_0(\lambda)\Delta t)\right) \\ \overline{C}_d &= U_C^{-1}\left(\frac{\xi_d}{\beta}(\lambda - \eta_0(\lambda)\Delta t)\right) \\ \overline{W}^a_u &= V_W^{-1}\left(\frac{\xi_u}{\beta}(\lambda - \eta_0(\lambda)\Delta t)\right) \\ \overline{W}^a_d &= V_W^{-1}\left(\frac{\xi_d}{\beta}(\lambda - \eta_0(\lambda)\Delta t)\right). \end{aligned}$$

Since $W_0\Delta t = \overline{C}_0\Delta t + \overline{W}^a_0\Delta t$, we can find the initial wealth given λ such that

$$\Rightarrow W_0\Delta t = U_C^{-1}(\lambda)\Delta t + V_W^{-1}(\eta_0(\lambda)\Delta t)\Delta t \triangleq W_0(\lambda)\Delta t.$$

Thus, W_0 is expressed as a function of λ. Under inada conditions we can show that $W_0(\lambda)$ is one to one and from the set R^{++} of positive real numbers onto itself. Let us denote the inverse function as $\lambda(W_0)$. We now give a solution to the original problem. For a given W_0, the first order conditions provide optimal choices.

If there is a *bequest function*, $M(\cdot)$, the agent's problem is to

$$\max_{C_0,C_u,C_d,W^a_0,W^a_u,W^a_d} U(C_0)\Delta t + V(W^a_0)\Delta t + \beta E\left[U(C_1) + V(W^a_1) + M(W^a_1)\right]\Delta t$$

subject to

$$\begin{aligned} W_0\Delta t &= C_0\Delta t + \pi_u\xi_u(C_u + W^a_u)\Delta t + \pi_d\xi_d(C_d + W^a_d)\Delta t \\ W^a_0\Delta t &= \pi_u\xi_u(C_u + W^a_u)\Delta t + \pi_d\xi_d(C_d + W^a_d)\Delta t. \end{aligned}$$

Let us define $F(W^a_1) \triangleq V(W^a_1) + M(W^a_1)$. Then the *Lagrangian* is

$$\begin{aligned} \mathcal{L} &= U(C_0)\Delta t + V(W^a_0)\Delta t + \beta E\left[U(C_1) + F(W^a_1)\right]\Delta t \\ &\quad +\lambda\left\{W_0 - C_0 - \pi_u\xi_u(C_u + W^a_u) - \pi_d\xi_d(C_d + W^a_d)\right\}\Delta t \\ &\quad -\eta_0\Delta t\left\{W^a_0 - \pi_u\xi_u(C_u + W^a_u)\Delta t - \pi_d\xi_d(C_d + W^a_d))\Delta t\right\}. \end{aligned}$$

Then the FOCs are

$$\begin{aligned}
C_0 &: U_C(\overline{C}_0)\Delta t = \lambda \Delta t \\
C_u &: U_C(\overline{C}_u)\Delta t = \frac{\xi_u}{\beta}(\lambda - \eta_0 \Delta t)\Delta t \\
C_d &: U_C(\overline{C}_d)\Delta t = \frac{\xi_d}{\beta}(\lambda - \eta_0 \Delta t)\Delta t \\
W_0^a &: V_W(\overline{W}_0^a)\Delta t = \eta_0 \Delta t \\
W_u^a &: F_W(\overline{W}_u^a)\Delta t = \frac{\xi_u}{\beta}(\lambda - \eta_0 \Delta t)\Delta t \\
W_d^a &: F_W(\overline{W}_d^a)\Delta t = \frac{\xi_d}{\beta}(\lambda - \eta_0 \Delta t)\Delta t.
\end{aligned}$$

Therefore,

$$\begin{aligned}
U_C(\overline{C}_u) &= F_W(\overline{W}_u^a) \\
U_C(\overline{C}_d) &= F_W(\overline{W}_d^a).
\end{aligned}$$

Fix $\lambda > 0$,

$$\begin{aligned}
\frac{p_u}{R}\left[\overline{C}_u(\eta_0) + \overline{W}_u^a(\eta_0)\right]\Delta t + \frac{p_d}{R}\left[\overline{C}_d(\eta_0) + \overline{W}_d^a(\eta_0)\right]\Delta t &= \overline{W}_0^a \Delta t \\
\Leftrightarrow V_W\left(\frac{p_u}{R}\left[\overline{C}_u(\eta_0) + \overline{W}_u^a(\eta_0)\right] + \frac{p_d}{R}\left[\overline{C}_d(\eta_0) + \overline{W}_d^a(\eta_0)\right]\right) &= V_W(\overline{W}_0^a) = \eta_0. \quad (3.17)
\end{aligned}$$

Given $\lambda > 0$, let us denote the solution of (3.17) by $\eta_0(\lambda)$. Then,

$$\begin{aligned}
\overline{C}_u &= U_C^{-1}\left(\frac{\xi_u}{\beta}(\lambda - \eta_0(\lambda))\right) \\
\overline{C}_d &= U_C^{-1}\left(\frac{\xi_d}{\beta}(\lambda - \eta_0(\lambda))\right) \\
\overline{W}_u^a &= F^{-1}\left(\frac{\xi_u}{\beta}(\lambda - \eta_0(\lambda))\right) \\
\overline{W}_d^a &= F^{-1}\left(\frac{\xi_d}{\beta}(\lambda - \eta_0(\lambda))\right).
\end{aligned}$$

Since $W_0 \Delta t = \overline{C}_0 \Delta t + \overline{W}_0^a \Delta t$,

$$\Rightarrow W_0 \Delta t = U_C^{-1}(\lambda)\Delta t + V_W^{-1}(\eta_0(\lambda))\Delta t \triangleq W_0(\lambda)\Delta t.$$

Therefore, we have essentially the same result as the case of no bequest utility.

3.2 A Three Period Model

In a three period model, we have

$$W_0 \Delta t = C_0 \Delta t + W_0^a \Delta t \left\langle \begin{array}{l} W_u \Delta t = C_u \Delta t + W_u^a \Delta t \left\langle \begin{array}{l} W_{uu}\Delta t = C_{uu}\Delta t + W_{uu}^a \Delta t \\ W_{ud}\Delta t = C_{ud}\Delta t + W_{ud}^a \Delta t \end{array}\right. \\ W_d \Delta t = C_d \Delta t + W_d^a \Delta t \left\langle \begin{array}{l} W_{du}\Delta t = C_{du}\Delta t + W_{du}^a \Delta t \\ W_{dd}\Delta t = C_{dd}\Delta t + W_{dd}^a \Delta t. \end{array}\right. \end{array}\right.$$

The agent's problem is to

$$\max_{C_0,C_1,C_2,W_0^a,W_1^a,W_2^a} U(C_0)\Delta t+V(W_0^a)\Delta t+\beta E\left[U(C_1)+V(W_1^a)\right]\Delta t+\beta^2 E\left[U(C_2)+V(W_2^a)\right]\Delta t$$

subject to

$$\begin{aligned}
W_0\Delta t &= C_0\Delta t+\pi_u\xi_u C_u\Delta t+\pi_d\xi_d C_d\Delta t+\pi_{uu}\xi_{uu}(C_{uu}+W_{uu}^a)\Delta t+\cdots\\
&\quad +\pi_{du}\xi_{du}(C_{du}+W_{du}^a)\Delta t+\pi_{dd}\xi_{dd}(C_{dd}\Delta t+W_{dd}^a)\Delta t\\
W_0^a\Delta t &= \pi_u\xi_u(C_u+W_u^a)\Delta t+\pi_d\xi_d(C_d+W_d^a)\Delta t+\pi_{uu}\xi_{uu}\left(C_{uu}+W_{uu}^a\right)\Delta t+\cdots\\
&\quad +\pi_{du}\xi_{du}(C_{du}+W_{du}^a)\Delta t+\pi_{dd}\xi_{dd}(C_{dd}+W_{dd}^a)\Delta t\\
\xi_u W_u^a\Delta t &= \pi_u\xi_{uu}\left(C_{uu}+W_{uu}^a\right)\Delta t+\pi_d\xi_{ud}\left(C_{ud}+W_{ud}^a\right)\Delta t\\
\xi_d W_d^a\Delta t &= \pi_u\xi_{du}\left(C_{du}+W_{du}^a\right)\Delta t+\pi_d\xi_{dd}\left(C_{dd}+W_{dd}^a\right)\Delta t.
\end{aligned}$$

The Lagrangian is

$$\begin{aligned}
\mathcal{L} &= \left\{U(C_0)+V(W_0^a)+\beta E\left[U(C_1)+V(W_1^a)\right]+\beta^2 E\left[U(C_2)+V(W_2^a)\right]\right\}\Delta t\\
&\quad +\lambda\Delta t\left[W_0-C_0-\pi_u\xi_u C_u-\pi_d\xi_d C_d-\pi_{uu}\xi_{uu}(C_{uu}+W_{uu}^a)-\cdots-\pi_{dd}\xi_{dd}(C_{dd}+W_{dd}^a)\right]\\
&\quad -\eta_0\Delta t\left[W_0^a-\pi_u\xi_u(C_u+W_u^a)\Delta t-\pi_d\xi_d(C_d+W_d^a)\Delta t\right.\\
&\qquad\qquad \left.-\pi_{uu}\xi_{uu}(C_{uu}+W_{uu}^a)\Delta t-\cdots-\pi_{dd}\xi_{dd}(C_{dd}+W_{dd}^a)\Delta t\right]\\
&\quad -\pi_u\eta_1^u\Delta t\left[\xi_u W_u^a-\pi_u\xi_{uu}\left(C_{uu}+W_{uu}^a\right)\Delta t-\pi_d\xi_{ud}\left(C_{ud}+W_{ud}^a\right)\Delta t\right]\\
&\quad -\pi_d\eta_1^d\Delta t\left[\xi_d W_d^a-\pi_u\xi_{du}\left(C_{du}+W_{du}^a\right)\Delta t-\pi_d\xi_{dd}\left(C_{dd}+W_{dd}^a\right)\Delta t\right].
\end{aligned}$$

Then the FOCs are as follows.

$$\left\{\begin{array}{llll}
C_0 & : U_C(\overline{C}_0)\Delta t & = & \lambda\Delta t\\
C_u & : U_C(\overline{C}_u)\Delta t & = & \frac{\xi_u}{\beta}\left(\lambda-\eta_0\Delta t\right)\Delta t\\
C_d & : U_C(\overline{C}_d)\Delta t & = & \frac{\xi_d}{\beta}\left(\lambda-\eta_0\Delta t\right)\Delta t\\
C_{uu} & : U_C(\overline{C}_{uu})\Delta t & = & \frac{\xi_{uu}}{\beta^2}\left(\lambda-\eta_0\Delta t-\eta_1^u\Delta t\right)\Delta t\\
C_{ud} & : U_C(\overline{C}_{ud})\Delta t & = & \frac{\xi_{ud}}{\beta^2}\left(\lambda-\eta_0\Delta t-\eta_1^u\Delta t\right)\Delta t\\
C_{du} & : U_C(\overline{C}_{du})\Delta t & = & \frac{\xi_{du}}{\beta^2}\left(\lambda-\eta_0\Delta t-\eta_1^d\Delta t\right)\Delta t\\
C_{dd} & : U_C(\overline{C}_{dd})\Delta t & = & \frac{\xi_{dd}}{\beta^2}\left(\lambda-\eta_0\Delta t-\eta_1^d\Delta t\right)\Delta t\\
W_0^a & : V_W(\overline{W}_0^a)\Delta t & = & \eta_0\Delta t\\
W_u^a & : V_W(\overline{W}_u^a)\Delta t & = & \frac{\xi_u}{\beta}\left(-\eta_0\Delta t+\eta_1^u\right)\Delta t\\
W_d^a & : V_W(\overline{W}_d^a)\Delta t & = & \frac{\xi_d}{\beta}\left(-\eta_0\Delta t+\eta_1^d\right)\Delta t\\
W_{uu}^a & : V_W(\overline{W}_{uu}^a)\Delta t & = & \frac{\xi_{uu}}{\beta^2}\left(\lambda-\eta_0\Delta t-\eta_1^u\Delta t\right)\Delta t\\
W_{ud}^a & : V_W(\overline{W}_{ud}^a)\Delta t & = & \frac{\xi_{ud}}{\beta^2}\left(\lambda-\eta_0\Delta t-\eta_1^u\Delta t\right)\Delta t\\
W_{du}^a & : V_W(\overline{W}_{du}^a)\Delta t & = & \frac{\xi_{du}}{\beta^2}\left(\lambda-\eta_0\Delta t-\eta_1^d\Delta t\right)\Delta t\\
W_{dd}^a & : V_W(\overline{W}_{dd}^a)\Delta t & = & \frac{\xi_{dd}}{\beta^2}\left(\lambda-\eta_0\Delta t-\eta_1^d\Delta t\right)\Delta t
\end{array}\right.$$

Fix $\lambda>0$, then

$$\begin{aligned}
\overline{W}_0^a\Delta t &= \frac{p_u}{R}\left[\overline{C}_u(\eta_0)+\overline{W}_u^a(\eta_0,\eta_1^u)\right]\Delta t+\frac{p_d}{R}\left[\overline{C}_d(\eta_0)+\overline{W}_d^a(\eta_0,\eta_1^d)\right]\Delta t\\
\overline{W}_u^a\Delta t &= \frac{p_u}{R}\left[\overline{C}_{uu}(\eta_0,\eta_1^u)+\overline{W}_{uu}^a(\eta_0,\eta_1^u)\right]\Delta t+\frac{p_d}{R}\left[\overline{C}_{ud}(\eta_0,\eta_1^u)+\overline{W}_{ud}^a(\eta_0,\eta_1^u)\right]\Delta t\\
\overline{W}_d^a\Delta t &= \frac{p_u}{R}\left[\overline{C}_{du}(\eta_0,\eta_1^d)+\overline{W}_{du}^a(\eta_0,\eta_1^d)\right]\Delta t+\frac{p_d}{R}\left[\overline{C}_{dd}(\eta_0,\eta_1^d)+\overline{W}_{dd}^a(\eta_0,\eta_1^d)\right]\Delta t.
\end{aligned}$$

Thus we have three unknown functionals η_0, η_1^u, and η_1^d and three equations to solve given λ. Note that the FOCs imply that

$$
\begin{aligned}
& V_W \left\{ \frac{p_u}{R} \left[U_C^{-1} \left(\frac{\xi_u}{\beta} \left(\lambda - \eta_0(\lambda)\Delta t\right) \right) + V_W^{-1} \left(\frac{\xi_u}{\beta} \left(-\eta_0(\lambda)\Delta t + \eta_1^u(\lambda)\right) \right) \right] \right. \\
& \left. + \frac{p_d}{R} \left[U_C^{-1} \left(\frac{\xi_d}{\beta} \left(\lambda - \eta_0(\lambda)\Delta t\right) \right) + V_W^{-1} \left(\frac{\xi_d}{\beta} \left(-\eta_0(\lambda)\Delta t + \eta_1^d(\lambda)\right) \right) \right] \right\} \\
& = \eta_0 \\
& V_W \left\{ \frac{p_u}{R} \left[U_C^{-1} \left(\frac{\xi_{uu}}{\beta^2} \left(\lambda - \eta_0(\lambda)\Delta t - \eta_1^u(\lambda)\Delta t\right) \right) + V_W^{-1} \left(\frac{\xi_{uu}}{\beta^2} \left(\lambda - \eta_0(\lambda)\Delta t - \eta_1^u(\lambda)\Delta t\right) \right) \right] \right. \\
& \left. + \frac{p_d}{R} \left[U_C^{-1} \left(\frac{\xi_{ud}}{\beta^2} \left(\lambda - \eta_0(\lambda)\Delta t - \eta_1^u(\lambda)\Delta t\right) \right) + V_W^{-1} \left(\frac{\xi_{ud}}{\beta^2} \left(\lambda - \eta_0(\lambda)\Delta t - \eta_1^u(\lambda)\Delta t\right) \right) \right] \right\} \\
& = \frac{\xi_u}{\beta} \left(-\eta_0 \Delta t + \eta_1^u\right) \\
& V_W \left\{ \frac{p_u}{R} \left[U_C^{-1} \left(\frac{\xi_{du}}{\beta^2} \left(\lambda - \eta_0(\lambda)\Delta t - \eta_1^d(\lambda)\Delta t\right) \right) + V_W^{-1} \left(\frac{\xi_{du}}{\beta^2} \left(\lambda - \eta_0(\lambda)\Delta t - \eta_1^d(\lambda)\Delta t\right) \right) \right] \right. \\
& \left. + \frac{p_d}{R} \left[U_C^{-1} \left(\frac{\xi_{dd}}{\beta^2} \left(\lambda - \eta_0(\lambda)\Delta t - \eta_1^d(\lambda)\Delta t\right) \right) + V_W^{-1} \left(\frac{\xi_{dd}}{\beta^2} \left(\lambda - \eta_0(\lambda)\Delta t - \eta_1^d(\lambda)\Delta t\right) \right) \right] \right\} \\
& = \frac{\xi_d}{\beta} \left(-\eta_0 \Delta t + \eta_1^d\right).
\end{aligned}
$$

3.3 A Four Period Model

Similarly,

$$
\begin{aligned}
\mathcal{L} = & \left\{ U(C_0) + V(W_0^a) + \cdots + \beta^3 E\left[U(C_3) + V(W_3^a)\right] \right\} \Delta t \\
& + \lambda \Delta t \left\{ W_0 - C_0 - \pi_u \xi_u C_u - \cdots - \pi_{uu} \xi_{uu} C_{uu} - \cdots \pi_{uuu} \xi_{uuu} (C_{uuu} + W_{uuu}^a) - \cdots - \pi_{ddd} (C_{ddd} + W_{ddd}^a) \right\} \\
& - \eta_0 \Delta t \left\{ W_0^a - \pi_u \xi_u (C_u + W_u^a) \Delta t - \cdots - \pi_{uuu} \xi_{uuu} (C_{uuu} + W_{uuu}^a) \Delta t - \cdots - \pi_{ddd} \xi_{ddd} (C_{ddd} + W_{ddd}^a) \Delta t \right\} \\
& - \pi_u \eta_1^u \Delta t \left\{ \xi_u W_u^a - \pi_u \xi_{uu} \left(C_{uu} + W_{uu}^a\right) \Delta t - \cdots - \pi_{dd} \xi_{udd} (C_{udd} + W_{udd}) \Delta t \right\} \\
& - \pi_d \eta_1^d \Delta t \left\{ \xi_d W_d^a - \pi_u \xi_{du} \left(C_{du} + W_{du}^a\right) \Delta t - \cdots - \pi_{dd} \xi_{ddd} (C_{ddd} + W_{ddd}^a) \Delta t \right\} \\
& - \pi_{uu} \eta_2^{uu} \Delta t \left\{ \xi_{uu} W_{uu}^a - \pi_u \xi_{uuu} \left(C_{uuu} + W_{uuu}^a\right) \Delta t - \pi_d \xi_{uud} \left(C_{uud} + W_{uud}^a\right) \Delta t \right\} \\
& - \pi_{ud} \eta_2^{ud} \Delta t \left\{ \xi_{ud} W_{ud}^a - \pi_u \xi_{udu} \left(C_{udu} + W_{udu}^a\right) \Delta t - \pi_d \xi_{udd} \left(C_{udd} + W_{udd}^a\right) \Delta t \right\} \\
& - \pi_{du} \eta_2^{du} \Delta t \left\{ \xi_{du} W_{du}^a - \pi_u \xi_{duu} \left(C_{duu} + W_{duu}^a\right) \Delta t - \pi_d \xi_{dud} \left(C_{dud} + W_{dud}^a\right) \Delta t \right\} \\
& - \pi_{dd} \eta_2^{dd} \Delta t \left\{ \xi_{dd} W_{dd}^a - \pi_u \xi_{ddu} \left(C_{ddu} + W_{ddu}^a\right) \Delta t - \pi_d \xi_{ddd} \left(C_{ddd} + W_{ddd}^a\right) \Delta t \right\}
\end{aligned}
$$

The optimality conditions are the following.

$$
\begin{cases}
\overline{C}_0\Delta t & = & U_C^{-1}\left(\lambda\right)\Delta t \\
\overline{C}_u\Delta t & = & U_C^{-1}\left(\frac{\xi_u}{\beta}\left(\lambda-\eta_0\Delta t\right)\right)\Delta t \\
\overline{C}_d\Delta t & = & U_C^{-1}\left(\frac{\xi_d}{\beta}\left(\lambda-\eta_0\Delta t\right)\right)\Delta t \\
\overline{C}_{uu}\Delta t & = & U_C^{-1}\left(\frac{\xi_{uu}}{\beta^2}\left(\lambda-\eta_0\Delta t-\eta_1^u\Delta t\right)\right)\Delta t \\
\overline{C}_{ud}\Delta t & = & U_C^{-1}\left(\frac{\xi_{ud}}{\beta^2}\left(\lambda-\eta_0\Delta t-\eta_1^u\Delta t\right)\right)\Delta t \\
\quad\vdots & & \\
\overline{C}_{uuu}\Delta t & = & U_C^{-1}\left(\frac{\xi_{uuu}}{\beta^3}\left(\lambda-\eta_0\Delta t-\eta_1^u\Delta t-\eta_2^{uu}\Delta t\right)\right)\Delta t \\
\overline{C}_{uud}\Delta t & = & U_C^{-1}\left(\frac{\xi_{uud}}{\beta^3}\left(\lambda-\eta_0\Delta t-\eta_1^u\Delta t-\eta_2^{uu}\Delta t\right)\right)\Delta t \\
\quad\vdots & & \\
\overline{W}_0^a\Delta t & = & V_W^{-1}\left(\eta_0\right)\Delta t \\
\overline{W}_u^a\Delta t & = & V_W^{-1}\left(\frac{\xi_u}{\beta}\left(-\eta_0\Delta t+\eta_1^u\right)\right)\Delta t \\
\overline{W}_d^a\Delta t & = & V_W^{-1}\left(\frac{\xi_d}{\beta}\left(-\eta_0\Delta t+\eta_1^d\right)\right)\Delta t \\
\overline{W}_{uu}^a\Delta t & = & V_W^{-1}\left(\frac{\xi_{uu}}{\beta^2}\left(-\eta_0\Delta t-\eta_1^u\Delta t+\eta_2^{uu}\right)\right)\Delta t \\
\overline{W}_{ud}^a\Delta t & = & V_W^{-1}\left(\frac{\xi_{ud}}{\beta^2}\left(-\eta_0\Delta t-\eta_1^u\Delta t+\eta_2^{ud}\right)\right)\Delta t \\
\quad\vdots & & \\
\overline{W}_{uuu}^a\Delta t & = & V_W^{-1}\left(\frac{\xi_{uuu}}{\beta^2}\left(\lambda-\eta_0\Delta t-\eta_1^u\Delta t-\eta_2^{uu}\Delta t\right)\right)\Delta t \\
\overline{W}_{uud}^a\Delta t & = & V_W^{-1}\left(\frac{\xi_{uud}}{\beta^2}\left(\lambda-\eta_0\Delta t-\eta_1^u\Delta t-\eta_2^{uu}\Delta t\right)\right)\Delta t \\
\quad\vdots & &
\end{cases}
$$

The pricing formula gives

$$
\begin{aligned}
\overline{W}_0^a\Delta t &= \frac{p_u}{R}\left[\overline{C}_u+\overline{W}_u^a\right]\Delta t+\frac{p_d}{R}\left[\overline{C}_d+\overline{W}_d^a\right]\Delta t \\
\overline{W}_u^a\Delta t &= \frac{p_u}{R}\left[\overline{C}_{uu}+\overline{W}_{uu}^a\right]\Delta t+\frac{p_d}{R}\left[\overline{C}_{ud}+\overline{W}_{ud}^a\right]\Delta t \\
\overline{W}_d^a\Delta t &= \frac{p_u}{R}\left[\overline{C}_{du}+\overline{W}_{du}^a\right]\Delta t+\frac{p_d}{R}\left[\overline{C}_{dd}+\overline{W}_{dd}^a\right]\Delta t \\
\overline{W}_{uu}^a\Delta t &= \frac{p_u}{R}\left[\overline{C}_{uuu}+\overline{W}_{uuu}^a\right]\Delta t+\frac{p_d}{R}\left[\overline{C}_{uud}+\overline{W}_{uud}^a\right]\Delta t \\
\overline{W}_{ud}^a\Delta t &= \frac{p_u}{R}\left[\overline{C}_{udu}+\overline{W}_{udu}^a\right]\Delta t+\frac{p_d}{R}\left[\overline{C}_{udd}+\overline{W}_{udd}^a\right]\Delta t \\
\overline{W}_{du}^a\Delta t &= \frac{p_u}{R}\left[\overline{C}_{duu}+\overline{W}_{duu}^a\right]\Delta t+\frac{p_d}{R}\left[\overline{C}_{dud}+\overline{W}_{dud}^a\right]\Delta t \\
\overline{W}_{dd}^a\Delta t &= \frac{p_u}{R}\left[\overline{C}_{ddu}+\overline{W}_{ddu}^a\right]\Delta t+\frac{p_d}{R}\left[\overline{C}_{ddd}+\overline{W}_{ddd}^a\right]\Delta t.
\end{aligned}
$$

3.4 An Infinite Horizon Model

In infinite horizon, we assume the *steady states* which implies the "time homogeneous" situation. We approximates this problem with a two period model such that

$$W(t) = C(t)\Delta t + W^a(t) \left\langle \begin{array}{l} W_u(t+\Delta t) = C_u(t+\Delta t)\Delta t + W_u^a(t+\Delta t) \\ W_d(t+\Delta t) = C_d(t+\Delta t)\Delta t + W_d^a(t+\Delta t). \end{array} \right.$$

Note that now the wealth is in sense of stock. Then the pricing equation becomes

$$\begin{aligned} V_W \Bigg\{ \frac{p_u}{R} \left[U_C^{-1} \left(\frac{\xi_u}{\beta} (\lambda - \eta(\lambda)\Delta t) \right) \Delta t + V_W^{-1} \left(\eta \left(\frac{\xi_u}{\beta} (\lambda - \eta(\ \lambda)\Delta t) \right) \right) \right] \\ + \frac{p_d}{R} \left[U_C^{-1} \left(\frac{\xi_d}{\beta} (\lambda - \eta(\lambda)\Delta t) \right) \Delta t + V_W^{-1} \left(\eta \left(\frac{\xi_d}{\beta} (\lambda - \eta(\ \lambda)\Delta t) \right) \right) \right] \Bigg\} = \eta. \end{aligned}$$

Another difference with the two period model in finite horizon is that the arguments for the second period wealth utility is expressed as the argument of η again, which stems from the steady states for wealth. Then the optimality conditions are

$$\begin{aligned} \overline{C}_0 \Delta t &= U_C^{-1}(\lambda)\Delta t \\ \overline{C}_u \Delta t &= U_C^{-1} \left(\frac{\xi_u}{\beta} (\lambda - \eta\Delta t) \right) \Delta t \\ \overline{C}_d \Delta t &= U_C^{-1} \left(\frac{\xi_d}{\beta} (\lambda - \eta\Delta t) \right) \Delta t \\ \overline{W}_0^a &= V_W^{-1}(\eta) \\ \overline{W}_u^a &= V_W^{-1} \left(\eta \left(\frac{\xi_u}{\beta} (\lambda - \eta\Delta t) \right) \right) \\ \overline{W}_d^a &= V_W^{-1} \left(\eta \left(\frac{\xi_d}{\beta} (\lambda - \eta\Delta t) \right) \right). \end{aligned}$$

Since $W_0 = \overline{C}_0 \Delta t + \overline{W}_0^a$, $W_0 = U_C^{-1}(\lambda)\Delta t + V_W^{-1}(\eta)$.

4 Numerical Example

In this section, we show numerical examples for the multi-period problem stated above. We assume the following utility functions: the *CRRA* utility for consumption and the *HARA* utility for wealth such that

$$\begin{aligned} U(C) &= \frac{C^{1-\gamma_1}}{1-\gamma_1} \\ V(W^a) &= \varepsilon \cdot \frac{(W^a + A)^{1-\gamma_2}}{1-\gamma_2} \qquad (A > 0, \varepsilon > 0). \end{aligned}$$

Note that the problems becomes similar to the Merton problem as $\varepsilon \to 0$.

4.1 A Four Period Example

For the simplification of exposition, we remind the two period example for a while. With the two utility functions above, the pricing equation becomes

$$\varepsilon\left\{\frac{p_u}{R}\left[\left(\frac{\xi_u}{\beta}(\lambda-\eta_0(\lambda)\Delta t)\right)^{-\frac{1}{\gamma_1}}+\left(\frac{1}{\varepsilon}\left(\frac{\xi_u}{\beta}(\lambda-\eta_0(\lambda)\Delta t)\right)\right)^{-\frac{1}{\gamma_2}}-A\right]\right.$$
$$\left.+\frac{p_d}{R}\left[\left(\frac{\xi_d}{\beta}(\lambda-\eta_0(\lambda)\Delta t)\right)^{-\frac{1}{\gamma_1}}+\left(\frac{1}{\varepsilon}\left(\frac{\xi_d}{\beta}(\lambda-\eta_0(\lambda)\Delta t)\right)\right)^{-\frac{1}{\gamma_2}}-A\right]+A\right\}^{-\gamma_2}=\eta_0.$$

Even though the η_0 is a functional, we can try a numerical iteration to find the numerical value of η_0 if it converges to some finite numerical value. Then we need to set a candidate for the initial η_0. In Merton problem, the optimal consumption is affected by wealth and the marginal propensity to consume. Applying to our example,

$$\begin{aligned}
\Rightarrow\quad \frac{U_C^{-1}(\lambda)}{V_W^{-1}(\eta)} &= K\\
\Rightarrow\quad V_W^{-1}(\eta) &= \frac{1}{K}U_C^{-1}(\lambda)\\
\Rightarrow\quad \eta(\lambda) &= V_W\left(\frac{1}{K}U_C^{-1}(\lambda)\right)\\
&= \varepsilon\left(\frac{1}{K}\lambda^{-\frac{1}{\gamma_1}}+A\right)^{-\gamma_2}
\end{aligned}$$

where

$$K=r+\frac{\rho-r}{\gamma_1}+\frac{1}{2}\frac{\gamma_1(\gamma_1-1)}{\gamma_1^2}\theta^2$$

is the marginal propensity to consume and $\theta=\frac{\mu-r}{\sigma}$ is the *market price of the risk*. We use this η as the initial guess for η_0 given λ. We calculate the stock investment at period 0 as $\Delta=\frac{W_u-W_d}{u-d}$. With this implementation, we get the numerical values for the optimal solutions and the marginal utilities for a four period problem.

Before to examine the optimal solution, we need to verify whether the numerical value converges. Figure (1) shows that the η_0 converges for a four period example. We compare the optimal solutions with the ones from Merton problem. Figure (3) shows an intuitive result that an agent who has a direct utility from his wealth exhibits a decreasing marginal propensity to consume and an increasing marginal propensity to invest in the risky assets compared to the one from Merton problem. From Figure (4), we can see the optimal wealth at the 'best' state of the wealth dynamics at terminal period converges to zero as the number of period increases. This is an intuitive result since even though there is utility of wealth during the lifetime, the utility does not dominate at the end.

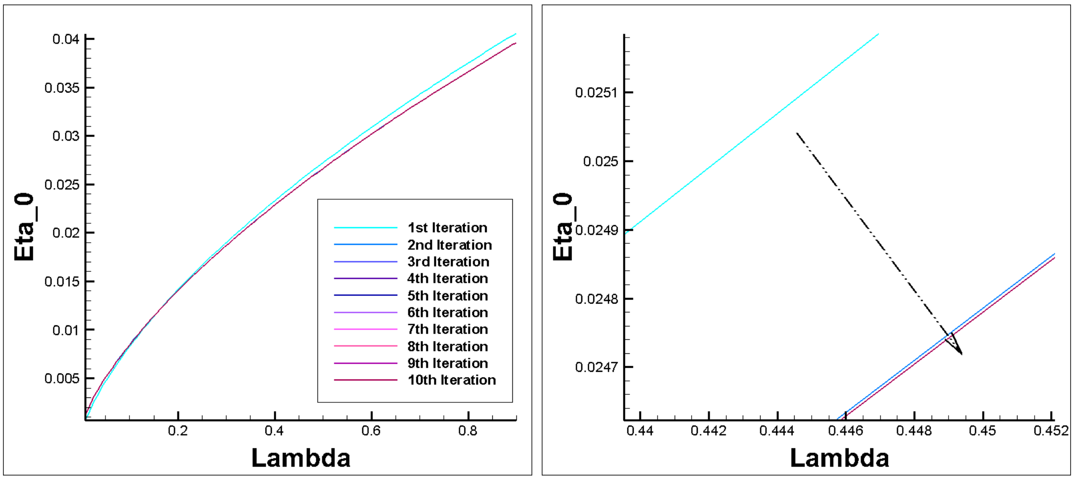

Figure 1: 4 period example, Convergence of η_0 by increasing iteration

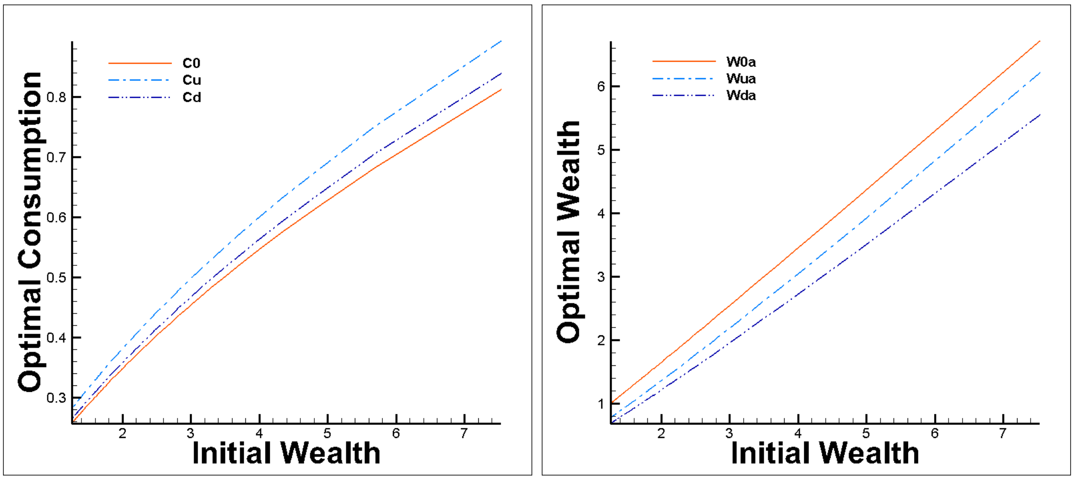

Figure 2: 4 period example, Optimal consumption and wealth for first two periods

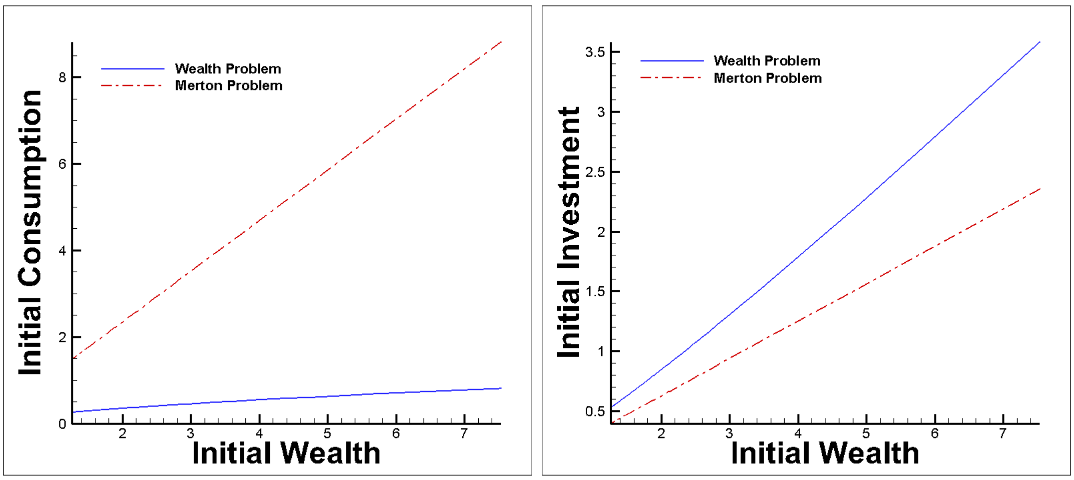

Figure 3: 4 period example, Comparison of initial consumption and initial investment with Merton problem

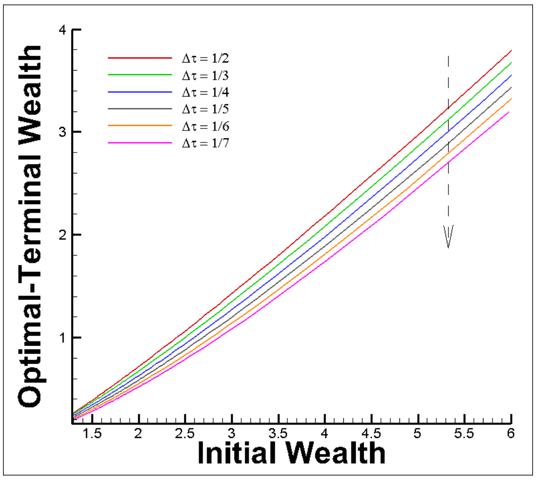

Figure 4: Convergence of optimal-terminal wealth to zero

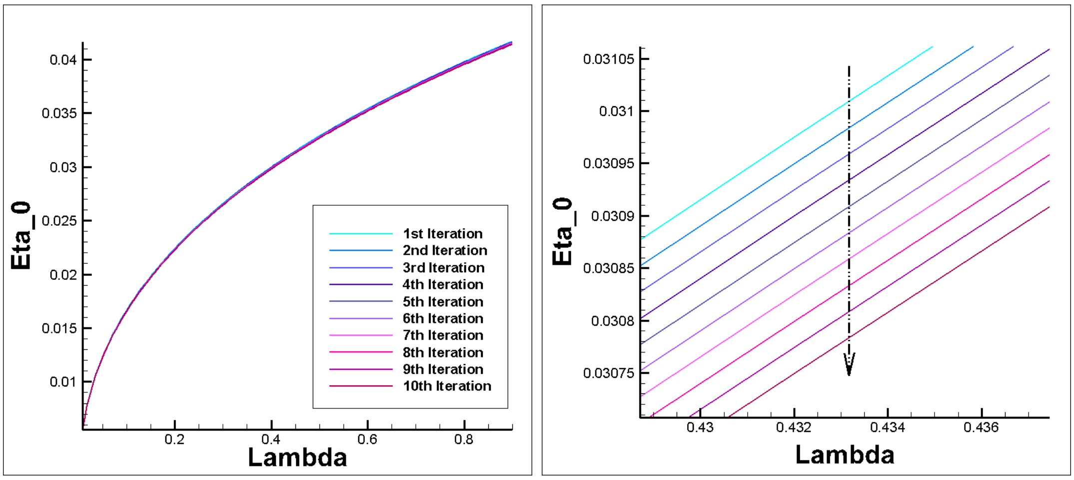

Figure 5: Infinite horizon example, Convergence of η by increasing iteration

4.2 Infinite Horizon Example

For an infinite horizon problem, we have

$$\varepsilon\left\{\frac{p_u}{R}\left[\left(\frac{\xi_u}{\beta}(\lambda-\eta(\lambda)\Delta t)\right)^{-\frac{1}{\gamma_1}}\Delta t+\left(\frac{1}{\varepsilon}\eta\left(\frac{\xi_u}{\beta}(\lambda-\eta(\lambda)\Delta t)\right)\right)^{-\frac{1}{\gamma_2}}-A\right]\right.$$
$$\left.+\frac{p_d}{R}\left[\left(\frac{\xi_d}{\beta}(\lambda-\eta(\lambda)\Delta t)\right)^{-\frac{1}{\gamma_1}}\Delta t+\left(\frac{1}{\varepsilon}\eta\left(\frac{\xi_d}{\beta}(\lambda-\eta(\lambda)\Delta t)\right)\right)^{-\frac{1}{\gamma_2}}-A\right]+A\right\}^{-\gamma_2}=\eta.$$

We use the same method to get the numerical η as the finite horizon problem. Note that the initial guess for η is same for both the finite horizon problem and the infinite horizon problem in our example since the utility function of consumption is CRRA utility function and there is no bequest function for the finite horizon problem. We refer to the literature for details.

Figure (5) shows the convergence of η. Figure (7) shows the same result as the one

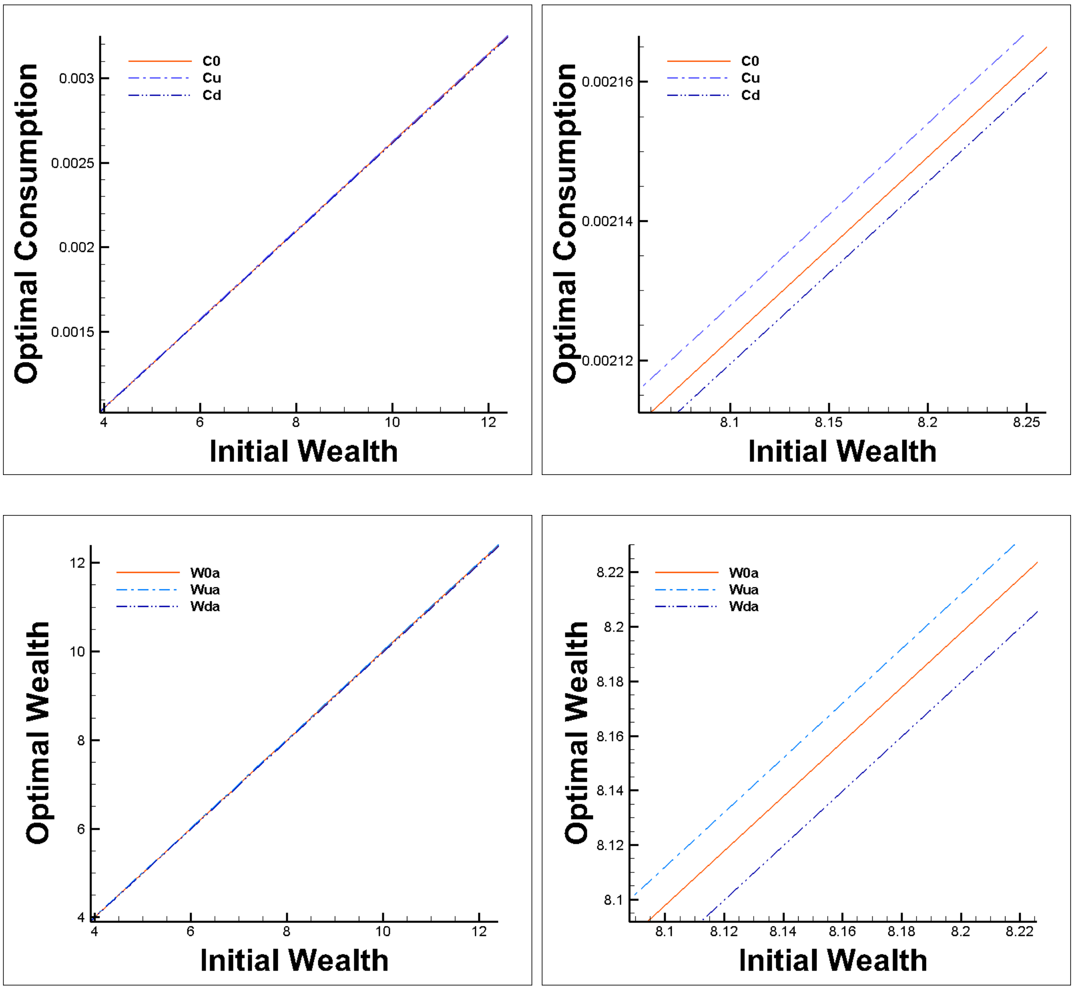

Figure 6: Infinite horizon example, Optimal consumption and wealth within two periods

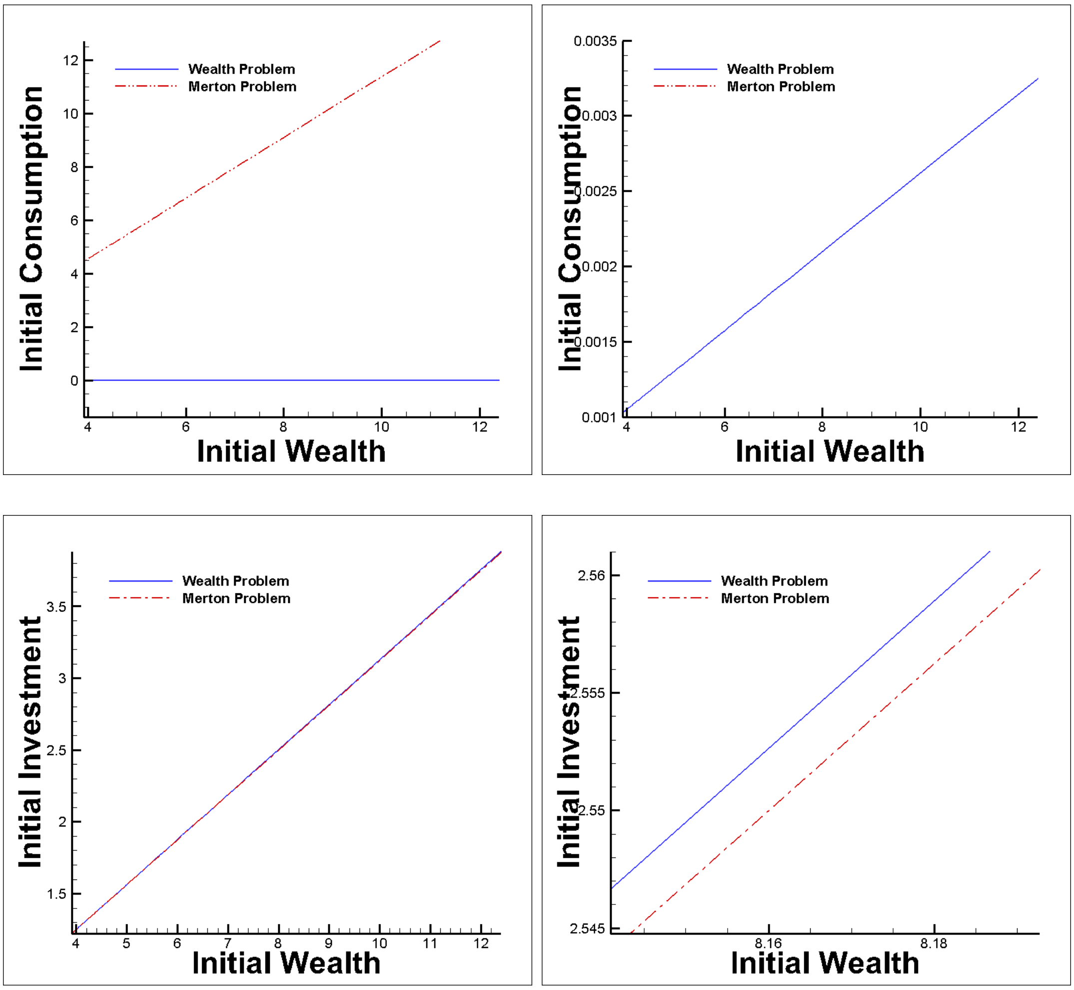

Figure 7: Infinite horizon example, Comparison of initial consumption and initial investment

from the finite horizon example.

5 Conclusion

In this study, we question the reason for the wealth accumulation. The accumulation is easily observed in the real world especially about *the rich*. We set up a continuous time optimization problem with a direct utility from wealth during the lifetime. We formulate the problem by using both *Stochastic Maximum Principle* and a *Martingale Method*, and we find the optimality conditions. We show the two approaches are equivalent to solve the problem. For numerical examples, we solve multi-period problems for finite horizon and infinite horizon by discrete time approximation. The results of our examples show that an agent whose a utility of wealth exhibits a *decreasing marginal propensity to consume(MPC)* and an *increasing marginal propensity to invest in the risky assets(MPIR)*, which is consistent with empirical evidence.

References

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