Liquidity Crashes and Robust Portfolio Management*

Bong-Gyu Jang**, Seungkyu Lee**, and Seyoung Park**

April 19, 2013.

Abstract

We find robust portfolio rules for ambiguity-aversive fund managers in a financial market with transaction costs. The model proposed in this paper permit liquidity premium much bigger than those found by most empirical literature. Using reasonably-calibrated parameters, we find liquidity premium obtained from the model is much bigger, so transaction costs can have a significant effect on investors' optimal investment behaviors. We also show that a high ambiguity aversion could be an explanation for a puzzling feature during economic crises that liquidity was greatly reduced in the financial market. Our model shows that a fund manager with a higher ambiguity aversion requires much bigger liquidity premium at times of down markets than at times of up markets.

**Department of Industrial and Management Engineering, POSTECH, Republic of Korea,

E-mail: bonggyujang@postech.ac.kr (Jang), lseungkyu@gmail.com (Lee), seyoungdog@postech.ac.kr (Park).

1 Introduction

> ... Bank of Canada Governor Mark Carney, said that global liquidity is about to dry up again as the European banking system deleverages, and warned that the real economy will soon feel the impact. "Global liquidity has fluctuated wildly over the past five years and we are on the cusp of another retrenchment," Mr. Carney said in the text of a speech, which was focused on global liquidity, to the Canada-U.K. Chamber of Commerce in London. Mr. Carney, who was appointed chairman of the Financial Stability Board at last week's G20 Summit, said market volatility is increasing and activity declining as global liquidity shrinks. (*The Wall Street Journal*, November 8, 2011)

The above quote from The Wall Street Journal implies that the global liquidity would dry up or shrink during the recent global financial crisis. The global crisis in 2008 has yielded many negative effects on the financial economy, such as liquidity crash[1], credit crunch etc. It seems that liquidity has been dried up in the subprime mortgage crisis; banks are reluctant to lend to individuals, firms, other banks, and capital market participants, and, thus, loan securitization is significantly reduced.(Berger and Bouwman 2009)

The liquidity dry-up might be followed by the fact that the exposure to liquidity risk could result in a negative externality (Acharya *et al.* 2011, Perotti and Suarez 2011), causing financial institutions have low liquidity. Researchers have been trying to explain the liquidity crash in terms of various aspects. The wealth effects arising from decreasing absolute risk aversion (Kyle and Xiong 2001), tighter financing constraints (Gromb and Vayanos 2002, Brunnemeier and Pedersen 2008), and strategic intermediation across segmented markets (Rahi and Zigrand 2007) would play a key role in explaining the liquidity crunch. More recently, Brunnermeier (2009) demonstrated that the lending boom prior to the global financial crisis in 2008 through loan securitization yielded that the asset holdings of banks increasingly consist of shorter maturity instruments. Thus, banks were exposed to the liquidity dry-up.

[1]In May 6, 2010, the prices of thousands of individual equity securities and exchange traded funds declined by significant amounts; at most more than 60%. This *flash crash* is also referred to as a liquidity crash, arising from the contagious illiquidity from sharply increased illiquidity of the S&P 500 futures simultaneously infecting other asset classes (Cespa and Foucault 2012).

Adverse selection for dealers in the days preceding the flash crash allowed high frequency market-makers to cut down on their liquidity provision, resulting in the liquidity dry-up during the crash (Easley *et al.* 2011). Cespa and Foucault (2012) decipher the liquidity crash as a switch from the equilibrium with high liquidity to the equilibrium with low liquidity without any clear reason. The liquidity requirement suggested by Basel III deteriorates market liquidity in the sense that the requirement decreases the future need to gather cash, and it might even lead to liquidity dry-up (Malherbe 2013). This paper provides an explanation for liquidity crash through ambiguity aversion of fund managers. We assume that fund managers are myopic (Gompers 1994, Marston and Craven 1998, Edmans 2009), because they are evaluated on their fund performance in the time horizon of a year or 6-months. We investigate the trading behaviors in the up markets and down markets and show how ambiguity-averse fund managers would cause the liquidity dry-up.

There are a few researches investigate the relationship between liquidity and ambiguity aversion of investors. Ambiguity aversion, in decision theory, refers an attitude of preference for known risks over unknown risks. When a decision maker knows the probability measure to guide choice, we classify the risk as *a known risk*. Ambiguity-averse decision makers are concerned about the unknown probability of the risk, not the risk itself.[2] When it comes to the relationship between liquidity and ambiguity of investors, Caballero and Krishnamurthy (2007) present that *Knightian uncertainty* is closely associated with liquidity hoarding. In their model, investors show the panic behavior such as flight to quality when Knightian uncertainty increases. Moreover, Routledge and Zin (2009) present a monopolistic market-maker's model displaying how liquidity crises are closely related to uncertainty aversion effect on the optimal portfolio. Recently, Easley and O'Hara (2010) develop a model in which illiquidity stems from uncertainty and they illustrate how uncertainty-averse investors are reluctant to participate in the stock market.

Our paper focuses on the mechanism of how ambiguity aversion induces liquidity crash

[2] Decision makers' ambiguity has been already investigated by several experimental tests; Ellsberg (1961), Kahneman and Tversky (1979), and Tversky and Kahneman (1992). They commonly assert that people's decision-making process cannot be explained by a simple rational expectation. Kahneman and Tversky (1979) propose prospect theory, which can explain decision makers' loss aversion. Knight (1921) and Keynes (1936) assert that economic decision makers are ambiguity averse. This is why we sometimes replace ambiguity with Knightian uncertainty.

during economic crises. This paper largely differs from the existing literature dealing with the relationship between liquidity and ambiguity aversion in a modeling point of view. Specifically, we incorporate ambiguity aversion of fund managers into the standard portfolio selection problem with transaction costs[3] and compute liquidity premium[4] of fund managers in the up markets and down markets. We show that a high ambiguity aversion could be an explanation for a puzzling feature during economic crises that liquidity was greatly reduced in the financial market. Our model shows that a fund manager with a higher ambiguity aversion requires much bigger liquidity premium at times of down markets than at time of up markets. This can be interpreted as that the manager who is very cautious about her estimates of the first moment of stock return would be reluctant to trade in the stock market in the down markets unless she obtains a significant compensation to trade in exchange for no-transaction costs. When an economic recession starts, the non-trading of ambiguity-averse fund manager could cause liquidity crash or liquidity dry-up.

Traditional portfolio selection with the unrealistic assumption of not considering investors' ambiguity failed to explain several financial puzzles observed in real market data[5]. One of the interesting financial puzzles is the *liquidity premium puzzle*. Traditional portfolio selection models with transaction costs (Constantinides 1979, 1986, Davis and Norman 1990, Liu and Loewenstein 2002) imply that the transaction costs have only a second-order effect on liquidity premium in contrast to what the evidences from empirical researches suggest. Specifically, liquidity premium to transaction cost (LPTC) ratio calculated from Constantinides'

[3]There are portfolio choice problems without transaction costs, taking ambiguity aversion into account. Gilboa and Schmeidler (1989) axiomatize max-min utility, which is developed into robustness theory. Hansen and Sargent (2001) and Hansen *et al.* (2006) extend Gilboa and Schmeidler's static problem into continuous time. Epstein and Schneider (2003) axiomatize an intertemporal version of multiple-priors utility, which is possible to be converted into a recursive structure for utility. These robust consumption and portfolio choice problems commonly assume that agents may fear model misspecification. The model misspecification is equivalent to Kightian uncertainty in that it occurs when decision makers do not know the exact probability measure. Because investors' estimated models are ambiguous, i.e. investors are not fully assured by their estimated, or calibrated, models, they make robust decision.

[4]Liquidity premium is defined as the decreasing amount equity premium due to the existence of transaction costs (Constantinides 1979, Jang *et al.* 2007).

[5]Mehra and Prescott (1985) argue that an immoderately high degree of risk aversion is required for a representative agent with a rational expectation utility to explain the high equity premium in the real financial markets, an inconsistency known as the *equity premium puzzle*. Weil (1989) reports that this excessively high risk aversion induces implausibly high risk free rate, known as the *risk free rate puzzle*. Shiller (1981) addresses that equity volatility is excessively high, compared to changes in the fundamental, causing the *excess volatility puzzle*. There are numerous financial abnormalities which are hard to explain with the standard rational expectation model, other than the aforementioned puzzle."

(1979) theoretical work is only about 0.07, while market-data-based LPTC ratio obtained from Amihud and Mendelson's (1986) empirical research is about 1.9.

There have been a lot of attempts to fill the gaps of LPTC between theory and empirical studies. Jang *et al.* (2007) show that transaction costs have first-order effect on liquidity premium in the presence of Markov regime-switching market environment. Lynch and Tan (2009) assert that LPTC ratio is much larger than that of Constantinides' (1979) model when they take return predictability into consideration. Dai *et al.* (2012) try to explain high market-based LPTC ratio by taking market closure effect into account. They focus on the effect of dynamism of investment opportunity set, which comes from the volatility change across trading and nontrading periods. Whereas most researchers are interested in dynamically fluctuating market conditions to resolve liquidity premium puzzle, our paper is distinguished from theirs in that it investigates the effect of ambiguity aversion of fund managers on liquidity premium under the constant investment opportunity set.

Along with these fundamental research works in robustness theory, various types of robustness have been developed to give answers to unsolved financial puzzles. Uppal and Wang (2003) develop a framework that allows different levels of ambiguity about any subset of multiple stocks, and explain the *under-diversification puzzle*. Garlappi *et al.* (2007) characterize ambiguity aversion via minimization over multiple priors which belong to a confidence interval around the estimated returns. Maenhout (2004) propose homothetic robustness that preserves wealth independence and analytical tractability, and show that robustness dramatically decreases the demand for equities, which may be an explanation for the equity premium puzzle.[6] Modifying Maenhout's (2004) homothetic robustness, Liu *et al.* (2005) investigate the asset pricing implication of imprecise knowledge about rare events, and find that ambiguity aversion related to rare events can explain the *option smirk*.

More recently, Zhu (2011) explains several stylized facts concerning catastrophe-linked securities premium spread, using modified Maenhout's (2004) homothetic robustness. Ju and Miao (2012) propose a generalized recursive smooth ambiguity model[7] which can match

[6]Maenhout (2004) also succeed in explaining the risk free rate puzzle.

[7]Klibanoff *et al.* (2005) and Klibanoff *et al.* (2011) first axiomatize smooth ambiguity model and Hansen and Sargent (2011) provide how to calibrate this smooth ambiguity model.

the mean equity premium, the mean risk-free rate, and the volatility of the equity premium observed in the data at a time.[8] These research works commonly argue that because the level of ambiguity aversion estimated from market data is considerably large, the significant gap between theory and market data such as the equity premium puzzle, risk free rate puzzle etc. can be filled when we replace risk aversion with the sum of two components: ambiguity aversion and risk aversion.

This paper is also motivated from the same logic: when we take ambiguity aversion as well as risk aversion into accounts, the theory-based LPTC ratio can be substantially elevated, so that it matches market-based LPTC ratio. Utilizing Maenhout's (2004) homothetic robustness we modify the standard portfolio selection problem with transaction costs for investors with constant relative risk aversion (CRRA) utility over terminal wealth. We find robust portfolio rules for ambiguity- aversive fund managers in a financial market with transaction costs. The model proposed in this paper permit liquidity premium much bigger than those found by most empirical literature. Using reasonably-calibrated parameters, we find liquidity premium obtained from the model is much bigger, so transaction costs can have a significant effect on investors' optimal investment behaviors.

This paper is organized as follows: Section 2 specifies a financial market and formulates robust portfolio problem with and without transaction costs. Section 3 provides analytical comparative statics about optimal trading strategy of investors. Section 4 presents numerical implications of optimal trading strategies, liquidity premium, and the relationship between liquidity crash and ambiguity aversion in the global crisis. Section 5 concludes the paper.

2 The Basic Model

2.1 The Financial Market

We consider a fund manager who wants to find the maximal score of her CRRA utility for terminal liquidation wealth at $T \in (0, \infty)$. The fund manager can trade two assets in a

[8] Their model also generates a variety of dynamic asset-pricing phenomena, including the procyclical variation of price-dividend ratios, the countercyclical variation of equity premia and equity volatility, the leverage effect, and the mean reversion of excess returns.

financial market: a bond and a stock. The bond price grows at a continuously compounded, constant rate r. On the other hand, the stock price, S_t, evolves by the relationship of

$$dS_t = \mu S_t dt + \sigma S_t dB_t,$$

where $\mu > 0$ is the expected rate of the stock return, $\sigma > 0$ is the stock volatility, and B_t is a standard one-dimensional Brownian motion.

The fund manager can buy the stock at the ask price, $S_t^A = (1+\alpha)S_t$, and sell the stock at the bid price, $S_t^B = (1-\beta)S_t$, where $\alpha \geq 0$ and $0 \leq \beta < 1$ denote the proportional transaction costs. We assume that the fund manager has the dollar amount invested in the bond, x_t, and the dollar amount invested in the stock, y_t. Then their evolutions are the following:

$$\begin{cases} dx_t = rx_t dt - (1+\alpha)dL_t + (1-\beta)dU_t, \\ dy_t = \mu y_t dt + \sigma y_t dB_t + dL_t - dU_t, \end{cases}$$

where L_t and U_t is the cumulative purchase and sale of the stock.

2.2 The Robust Portfolio Problem without Transaction Costs

The robust portfolio problem without transaction costs in this paper is similar to that in Maenhout (2004). The fund manager has CRRA preference and wants to maximize of her expected utility for terminal wealth w_T:

$$V^{RC}(0, w) = \max_{\pi} \mathbb{E}\left[\frac{{w_T}^{1-\gamma}}{1-\gamma}\right], \tag{1}$$

for constant relative risk aversion $\gamma > 0$, $\gamma \neq 1$, where π is the proportional amount invested in the stock. The wealth process of the fund manager w_t is subject to

$$dw_t = \{r + (\mu - r)\pi\}w_t dt + \sigma\pi w_t dB_t, \qquad \text{for } w_0 = w.$$

We suggest the robust portfolio rules of the fund manager without transaction costs in the following theorem.

Theorem 2.1. *Suppose that $\alpha = \beta = 0$. Then optimal risky investment π is given by*

$$\pi = \frac{\mu - r}{(\gamma + \theta)\sigma^2}.$$

Moreover, the value function is

$$V^{RC}(t, w) = f(t)\frac{w^{1-\gamma}}{1-\gamma},$$

where

$$f(t) = e^{a_0(T-t)}, \qquad \textit{where } a_0 = (1-\gamma)\Big(r + \frac{1}{2}\frac{1}{\gamma+\theta}\Big(\frac{\mu - r}{\sigma}\Big)^2\Big).$$

Proof. See Appendix 6.1.

Theorem 2.1 demonstrates that optimal risky investment π is exactly same with the classical Merton's risky investment except that risk-aversion γ is replaced by $\gamma + \theta$. The fund manager who worries about her imprecise estimates of the first moment of stock returns, $\theta > 0$, tends to invest less in the stock market than one without ambiguity aversion, $\theta = 0$. Notice that the fund manager who is not subject to transaction costs is myopic with respect to time t.

2.3 The Robust Portfolio Problem with Transaction Costs

In the presence of transaction costs, the robust portfolio problem of a fund mananger becomes

$$V(x, y, t) = \max_{(L,U)} \mathbb{E}\left[\frac{\big(x_T + (1-\beta)y_T\big)^{1-\gamma}}{1-\gamma}\right],$$

subject to

$$dx_t = rx_t dt - (1+\alpha)dL_t + (1-\beta)dU_t,$$

$$dy_t = \mu y_t dt + \sigma y_t dB_t + dL_t - dU_t.$$

Following Shreve and Soner (1994) and Dai and Yi (2009), we have the following HJB equation with homothetic robustness:

$$\begin{cases} \min\{-V_t - \mathcal{D}V - \sigma^2 y^2 h V_y - \dfrac{\sigma^2 y^2 h^2}{2\Psi(x,y,t)}, -(1-\beta)V_x + V_y, (1+\alpha)V_x - V_y\} = 0, \\ V(x,y,T) = \dfrac{\big(x + (1-\beta)y\big)^{1-\gamma}}{1-\gamma}, \end{cases} \tag{2}$$

where

$$\mathcal{D}V(x,y,t) = rxV_x + \mu y V_y + \frac{1}{2}\sigma^2 y^2 V_{yy},$$

$$V_t = \frac{\partial V}{\partial t}, \quad V_x = \frac{\partial V}{\partial x}, \quad V_y = \frac{\partial V}{\partial y}, \quad V_{yy} = \frac{\partial^2 V}{\partial y^2},$$

and

$$\Psi(x,y,t) = \frac{\theta}{(1-\gamma)V(x,y,t)}.$$

From the first order condition for h, we can obtain

$$\sigma^2 y^2 V_y + \frac{1}{\Psi} y^2 \sigma^2 h = 0,$$

or equivalently,

$$h = -\frac{\theta}{1-\gamma} \cdot \frac{V_y}{V}.$$

Consider the following transformation (Davis and Norman 1990, Shreve and Soner 1994) to reduce the dimension of our problem:

$$z \equiv \frac{x}{y}, \quad V(x,y,t) \equiv y^{1-\gamma}\varphi(z,t).$$

Then, the HJB equation (2) is written as

$$\begin{cases} \min\{-\varphi_t - \mathcal{D}^*\varphi, -(z+1-\beta)\varphi_z + (1-\gamma)\varphi, (z+1+\alpha)\varphi_z - (1-\gamma)\varphi\} = 0, \\ \varphi(z,T) = \dfrac{\big(z + (1-\beta)\big)^{1-\gamma}}{1-\gamma}. \end{cases}$$

where

$$\mathcal{D}^*\varphi = \frac{1}{2}\sigma^2 z^2 \varphi_{zz} + \big((r-\mu) + \sigma^2(\gamma+\theta)\big) z\varphi_z$$
$$+ \big((\mu - \frac{1}{2}\sigma^2(\gamma+\theta)\big)(1-\gamma)\varphi - \frac{1}{2}\sigma^2\theta\frac{1}{1-\gamma}z^2\frac{(\varphi_z)^2}{\varphi}.$$

Define

$$\psi(z,t) \equiv \frac{1}{1-\gamma}\ln[(1-\gamma)\varphi(z,t)].$$

It is easy to see that $\psi(z,t)$ satisfies

$$\begin{cases} \min\{-\psi_t - \mathcal{D}^{**}\psi, -(z+1-\beta)\psi_z + 1, (z+1+\alpha)\psi_z - 1\} = 0, \\ \psi(z,T) = \ln\big(z + (1-\beta)\big). \end{cases} \tag{3}$$

where

$$\mathcal{D}^{**}\psi = \frac{1}{2}\sigma^2 z^2 \psi_{zz} + \big((r-\mu) + \sigma^2(\gamma+\theta)\big) z\psi_z + \big((\mu - \frac{1}{2}\sigma^2(\gamma+\theta)\big)$$
$$+ \frac{1}{2}\sigma^2(1-\gamma-\theta)z^2(\psi_z)^2.$$

The closed form solution of the HJB equation (3) does not seem to be available. Therefore, we are required to develop numerical procedure to solve the equation. According to Dai and Zhong (2010), we use a finite difference scheme to discretize the HJB equation (3) and then apply a penalty method:

$$-\psi_t - \mathcal{D}^{**}\psi = K[(z+1-\beta)\psi_z - 1]^+ + K[-(z+1+\alpha)\psi_z - 1],$$

with terminal condition $\psi(z,T) = \ln\big(z + (1-\beta)\big)$, where K is chosen to be sufficiently large.

3 Analytical Comparative Statics

We can transform the HJB equation (3) into a double obstacle problem (Dai and Yi 2009). $\psi(z,t)$ satisfies the following parabolic double obstacle problem:

$$\begin{aligned}
&-\psi_t - \mathcal{D}^{**}\psi = 0, \ \text{ if } \frac{1}{z+1+\alpha} < \psi_z(z,t) < \frac{1}{z+1-\beta}, \\
&-\psi_t - \mathcal{D}^{**}\psi \geq 0, \ \text{ if } \psi_z(z,t) = \frac{1}{z+1+\alpha}, \\
&-\psi_t - \mathcal{D}^{**}\psi \leq 0, \ \text{ if } \psi_z(z,t) = \frac{1}{z+1-\beta}, \\
&\psi(z,T) = \ln\big(z+(1-\beta)\big), \quad z \in (\beta-1,+\infty).
\end{aligned}$$

Then the sell region, buy region, and no-transaction region are represented by the following:

$$\begin{aligned}
&\mathbf{SR} = \{(z,t) : \psi_z(z,t) = \frac{1}{z+1-\beta}\}, \\
&\mathbf{BR} = \{(z,t) : \psi_z(z,t) = \frac{1}{z+1+\alpha}\}, \\
&\mathbf{NT} = \{(z,t) : \frac{1}{z+1+\alpha} < \psi_z(z,t) < \frac{1}{z+1-\beta}\}.
\end{aligned}$$

We provide the theorem characterizes optimal behaviors of sell and buy boundaries by time-varying functions of $z_s(t)$ and $z_b(t)$, respectively.

Theorem 3.1. *There are two monotonically increasing functions $z_s(t) : [0,T) \to (\beta-1,+\infty)$ and $z_b(t) : [0,T) \to (\beta-1,+\infty)$ such that $z_s(t) < z_b(t)$ for all $t \in [0,T)$ and*

$$\boldsymbol{SR} = \{(z,t) : z \leq z_s(t), \ t \in [0,T)\}, \tag{4}$$

$$\boldsymbol{BR} = \{(z,t) : z \geq z_b(t), \ t \in [0,T)\}, \tag{5}$$

$$\boldsymbol{NT} = \{(z,t) : z_s(t) < z(t) < z_b(t), \ t \in [0,T)\}.$$

Moreover,

$$\begin{aligned}
&z_s(t) \leq z_s(T^-) = (1-\beta) z_M^{\theta}, \\
&z_b(t) \geq (1+\alpha) z_M^{\theta},
\end{aligned} \tag{6}$$

for all $t \in [0,T)$, where z_M^θ, ambiguity-adjusted Merton line, is given as

$$z_M^\theta = -\frac{\mu - r - \sigma^2(\gamma+\theta)}{\mu - r}. \tag{7}$$

Further, there exists a $t_b = T - \dfrac{1}{(\mu-r)} \ln\left(\dfrac{1+\alpha}{1-\beta}\right)$ such that

$$z_b(t) = +\infty, \text{ if and only if } t \in [t_b, T). \tag{8}$$

Proof. See Appendix 3.1.

Theorem 3.1 implies that it is optimal for fund managers to sell (buy) a stock if z, ratio of the dollar amount invested in a bond to the dollar amount in a stock, is lower (larger) than or equal to time-dependent functions $z_s(t)$ ($z_b(t)$). Further, the optimal stock trading strategy for the fund manager is to transact the minimum amount to keep the ratio z in the no-transaction region (Davis and Norman 1990, Liu and Loewenstein 2002, Dai *et al.* 2012). The fund managers become increasingly compelled to sell the stock as time approaches maturity due to the monotonic property of the sell and buy boundaries and imminent liquidation at maturity. Accordingly, the fund managers would no longer purchase the stock near the maturity, which is represented in (8).

The upper and lower bounds of the sell and buy boundaries are given in (6). The bounds depend on transaction costs α, β and *ambiguity-adjusted Merton line* z_M^θ in (7). When $\theta = 0$, z_M^0 becomes the classical Merton line (Davis and Norman 1990, Liu and Lowewnstein 2002). A larger ambiguity aversion θ drives up ambiguity-adjusted Merton line z_M^θ, and, thus, optimal risky investment becomes lower without transaction costs. The bounds of the sell and buy boundaries yield the width of the no-transaction region

$$(\alpha + \beta) z_M^\theta.$$

The width of the no-transaction region widens as ambiguity-adjusted Merton line z_M^θ increases, or equivalently, ambiguity aversion θ rises. Intuitively, fund managers with higher ambiguity aversion, which indicates higher risk aversion, would be unwilling to trade in order to avoid taking risks in stock market. Moreover, since fund managers would be reluctant to trade a stock with higher transaction costs, a larger transaction cost extends the width of the no-transaction region. Hence, it would be optimal for fund managers with higher ambiguity aversion and transaction costs to reduce their number of trading, and this is reflected in the fact that the width of the no-transaction region enlarges with respect to ambiguity aversion and transaction costs.

Let $\delta_\theta = (\mu - r - \sigma^2(\gamma+\theta))$ be the *variance-adjusted risk premium* with risk and ambiguity aversion. Without ambiguity aversion, $\theta = 0$, and in the case of logarithm utility, $\gamma = 1$, δ_0 becomes the variance-adjusted risk premium in Dai *et al.* (2012). Notice that investment opportunity becomes favorable (unfavorable) to fund managers as variance-adjusted risk premium δ_θ rises (falls). We introduce the theorem related to properties of optimal sell and buy boundaries with respect to δ_θ.

Theorem 3.2. *We have the following properties of optimal sell and buy boundaries:*

(1) If $\delta_\theta \le 0$, then $z_s(t) > 0$ for all t.

(2) If $\delta_\theta > 0$, then $z_s(t) < 0$ for all t, $z_b(t) < 0$ for $t < \hat{t}_b$ and $z_b(t) > 0$ for $t \ge \hat{t}_b$, where

$$\hat{t}_b = T - \frac{1}{\delta_\theta} \ln\Big(\frac{1-\beta}{1+\alpha}\Big).$$

Proof. See Appendix 6.3.

Theorem 3.2 shows that if the variance-adjusted risk premium with risk and ambiguity aversion, δ_θ, is lower than or equal to zero, then optimal sell boundary $z_s(t)$ is always positive for all t. Non-positive δ_θ is equivalent to an optimal risky investment π^* without transaction costs. In this case, the fraction of wealth invested in a stock, is lower than or equal to zero. Thus, leverage must not be optimal for fund managers without transaction costs. This could be applicable to them with transaction costs, which is represented by $z_s(t) > 0$ for all t. When $\delta_\theta > 0$, or equivalently, optimal risky investment π^* is larger than one, it is optimal for fund managers without transaction costs to take leverage. Further, the result might be also true for fund managers with transaction costs. Indeed, $z_b(t) < 0$ for $t < \hat{t}_b$ implies that the fund managers are willing to take leverage when there is a long time to maturity and they tend

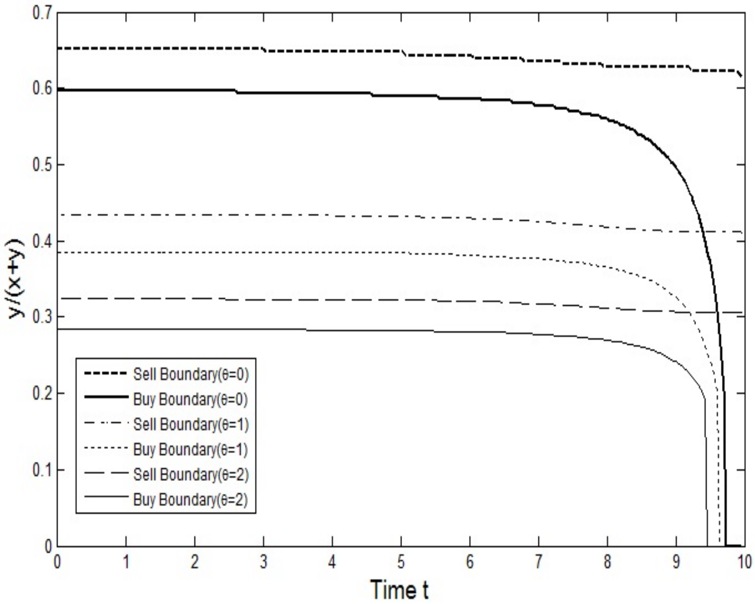
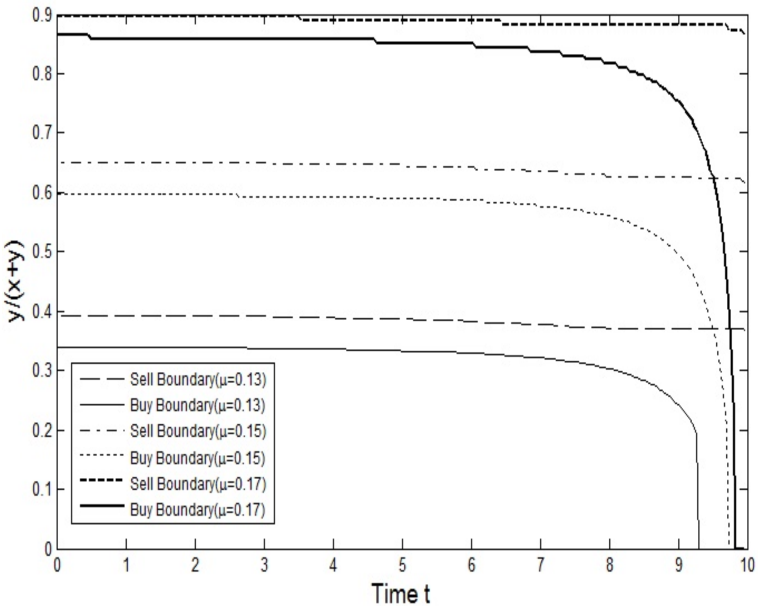


Figure 1: **The optimal trading strategy for the stock as a function of time** t: $r = 0.10$, $\mu = 0.15$, $\sigma = 0.2$, $\gamma = 2$, $T = 10$, $\alpha = 0.005$, $\beta = 0.005$ are used for parameter values.

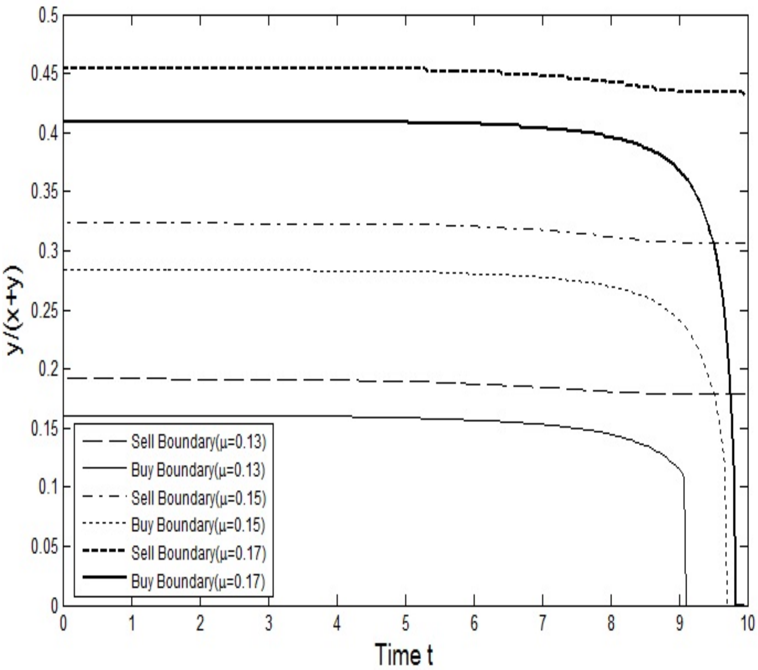
to deleverage near the maturity which is denoted by $z_b(t) > 0$ for $t \geq \hat{t}_b$. Note that since ambiguity aversion θ might play a key role in determining whether the sign of δ_θ is positive or not, the fund managers should consider ambiguity importantly in leverage management.

4 Implications

In this section, we explore the behaviors of optimal sell and buy boundaries with respect to ambiguity aversion θ. For a better understanding, we illustrate the boundaries utilizing $y/(x+y)$, the fraction of wealth invested in the stock, instead of z and explore optimal trading strategies through it. Note that if the amount of risky investment to wealth ratio is higher (lower) than some threshold, then it is optimal for fund managers to sell (buy) the stock until the ratio equals to the threshold. In addition, we investigate the impact of ambiguity aversion on the liquidity premia. Finally, we examine the relationship between liquidity crash during the global crisis and ambiguity. For this analysis, we exploit a finite difference scheme and a penalty method to solve a differential equation in Section 2.3.



(A) θ=0



(B) θ=2

Figure 2: **The optimal trading strategy for the stock as a function of time** t: $r = 0.10$, $\mu = 0.15$, $\sigma = 0.2$, $\gamma = 2$, $T = 10$, $\alpha = 0.005$, $\beta = 0.005$ are used for parameter values.

4.1 Baseline Parameters

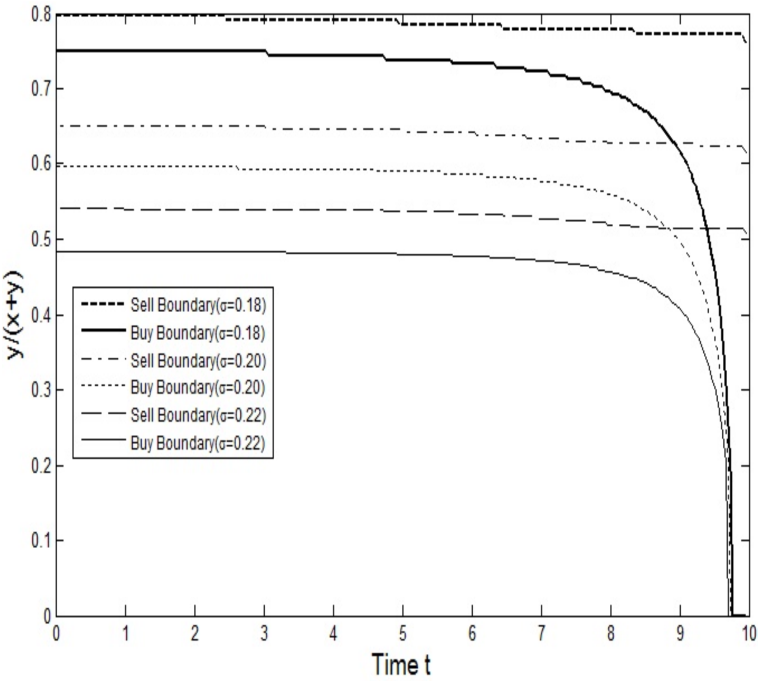
We set the following default parameter values: $r = 0.10$, $\mu = 0.15$, $\sigma = 0.2$, $\gamma = 2$, $T = 10$, $\alpha = 0.005$, $\beta = 0.005$. These values are same as in Constantinides (1986).

4.2 Optimal Trading Strategies

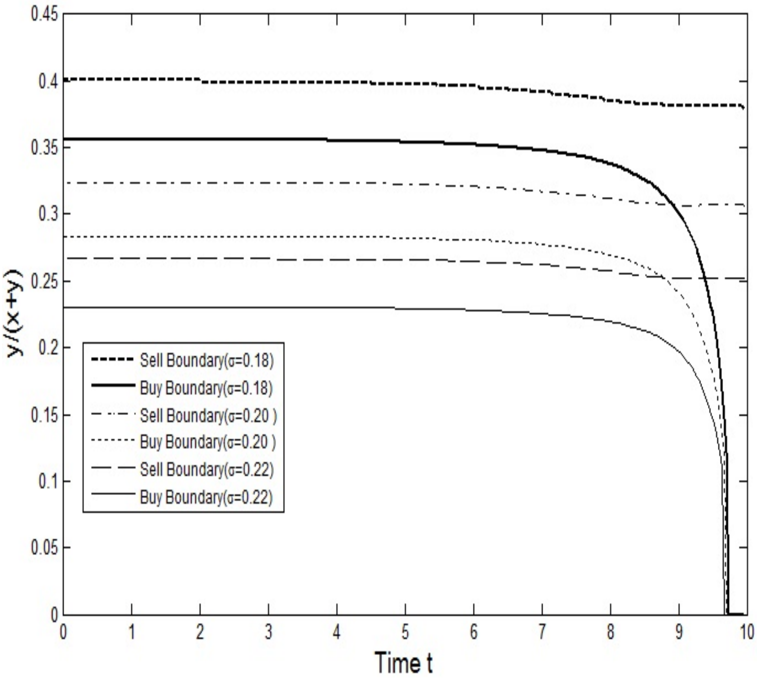
Figure 1 represents the optimal trading strategy for stock as a function of time t. The optimal sell and buy boundaries, $z_s(t)$ and $z_b(t)$, have monotonic increasing property of time t which can be easily inferred from $y/(x + y) = 1/(1 + z)$. The observation is consistent with the result in Theorem 3.1. As ambiguity aversion θ increases, the fraction of wealth invested in the stock decreases. In this sense, ambiguity aversion of fund managers might be an explanation of the *moderate equity holding puzzle*, which reflects the fact that the equity holdings of stock market participants are moderate (Gomes and Michaelides 2005).

Figure 2 denotes the optimal trading strategy for the stock as a function of time t for several values of expected rate of the stock return μ. A higher μ implies a larger fraction of wealth invested in the stock. A good investment opportunity would drive up the amount of optimal risky investment. Fund managers are reluctant to invest in the stock market when facing imprecise estimates of the first moment of stock returns, $\theta > 0$, and such behavior can be observed in Figure 2 (B). When $\theta = 2$ the amount of risky investment to wealth ratio is significantly lower than the one of $\theta = 0$ for all cases of μ. Specifically, the ratio of fund managers with ambiguity aversion is about two times lower than that of fund managers without ambiguity aversion. Fund managers with even moderate ambiguity aversion could remarkably reduce the fraction of wealth invested in stock even if the stock has a higher expected rate of return. The proportional amount invested in the stock under $\theta = 2$ and $\mu = 0.17$ is still lower than the one of $\theta = 0$ and $\mu = 0.15$. Fund managers may be more concerned about ambiguity than investment opportunity.

Figure 3 shows the optimal trading strategy for stock as a function of time t for several values of stock volatility σ. The investment opportunity worsens as stock volatility σ increases, so the fraction of wealth invested in the stock moves lower with respect to σ, which can be observed in both Figure 3 (A) and (B).



(A) θ=0



(B) θ=2

Figure 3: **The optimal trading strategy for the stock as a function of time** t: $r = 0.10$, $\mu = 0.15$, $\sigma = 0.2$, $\gamma = 2$, $T = 10$, $\alpha = 0.005$, $\beta = 0.005$ are used for parameter values.

The sensitivity of the optimal trading boundaries for various γ, risk aversion, is illustrated in Figure 4. Without ambiguity aversion, $\theta = 0$, the risky investment to wealth ratio shifts downward as risk aversion increases (Figure 4 (A)). Further, ambiguity aversion accelerates this effect in that the ratio decreases (Figure 4 (B)). Figure 5 displays the optimal trading strategy for the stock as a function of time t for various transaction costs α and β. Notice that the width of no-transaction region widens as transaction costs increase. Moreover, the ambiguity aversion induces fund managers to be in the no-transaction region, and, thus, when $\theta = 2$ the no-transaction region is remarkably wider than when $\theta = 0$.

4.3 Liquidity Premium

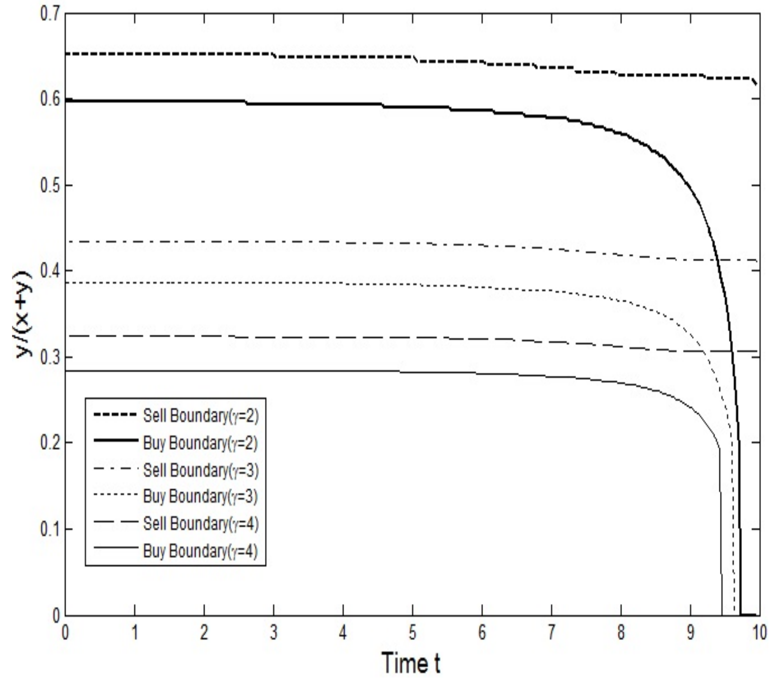
In this section, we reveal the great impact of fund managers' ambiguity aversion on liquidity premium, which can explain the well-known liquidity premium puzzle.

Liquidity premium is defined as the required amount of equity premium to compensate for transaction costs. We can formally define the liquidity premium as following:

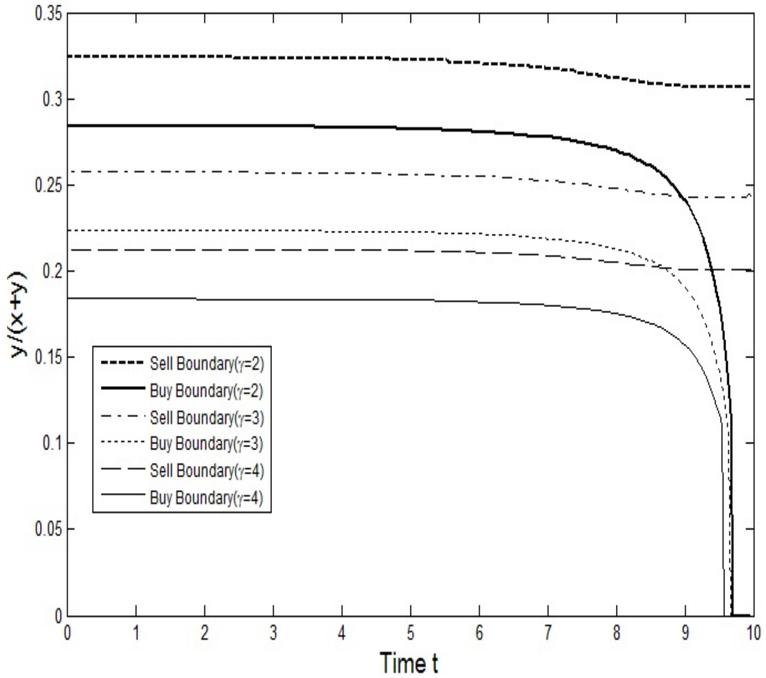
Definition 4.1. *Let Δ be the liquidity premium at $(z_M, 1)$. Then Δ is such that*

$$V(0, z_M, 1) = e^{aT}\frac{(z_M+1)^{1-\gamma}}{1-\gamma}, \qquad \textit{where } a = (1-\gamma)\Big(r + \frac{1}{2(\gamma+\theta)}\Big(\frac{\mu-\Delta-r}{\sigma}\Big)^2\Big).$$

Constantinides (1979) argue that LPTC ratio is about 0.06, which is too small when compared to empirical evidences. For instance, Amihud and Mendelson (1986) report that LPTC ratio is about 1.9 for NYSE stocks, much higher than what Constantinides (1979) suggest. Many researchers show that the inconsistency between theory and empirical studies stems from the fundamental assumption of Constantinides (1979): a risky asset follows a geometric Brownian motion without any other risk sources. Jang *et al.* (2007) introduce a stochastic investment opportunity set and calculate market-consistent LPTC ratio. They assume that stock volatility switches between two states of up-market and down-market using a Markov regime-switching model. Although they show that transaction costs have a first-order effect on liquidity premium under the stochastic investment opportunity set, their LPTC ratio is still quite small compared to market data. Our paper attempts to shed light on the

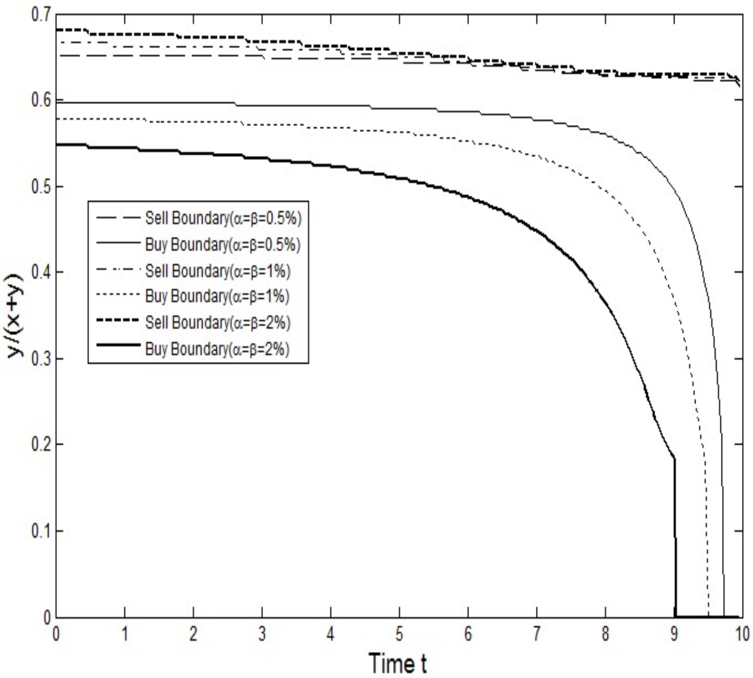


(A) θ=0

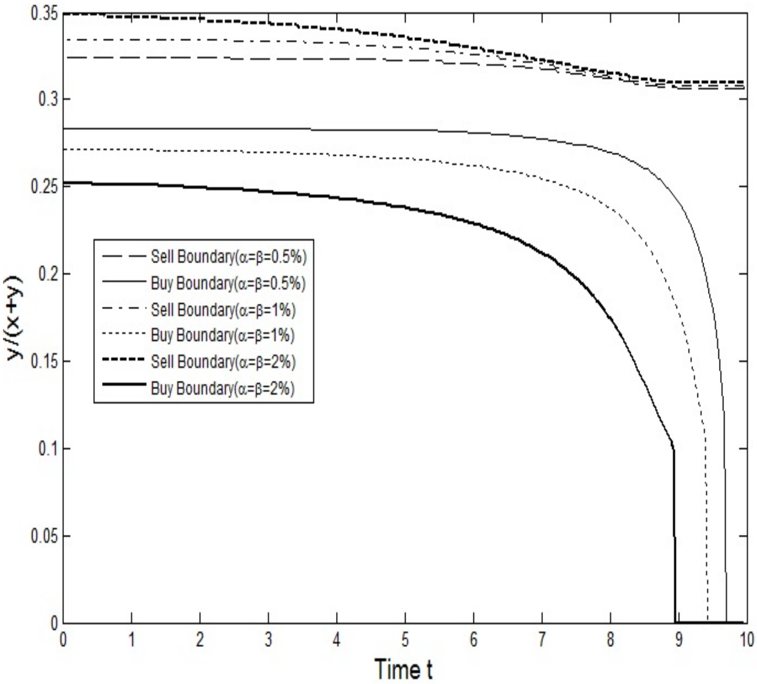


(B) θ=2

Figure 4: **The optimal trading strategy for the stock as a function of time** t: $r = 0.10$, $\mu = 0.15$, $\sigma = 0.2$, $\gamma = 2$, $T = 10$, $\alpha = 0.005$, $\beta = 0.005$ are used for parameter values.



(A) θ=0



(B) θ=2

Figure 5: **The optimal trading strategy for the stock as a function of time** t: $r = 0.10$, $\mu = 0.15$, $\sigma = 0.2$, $\gamma = 2$, $T = 10$, $\alpha = 0.005$, $\beta = 0.005$ are used for parameter values.

liquidity premium puzzle by incorporating ambiguity aversion of fund managers into the portfolio selection problem with transaction costs. Table 1 summarizes our numerical results

θ	z_s	z_b	LPTC ratio
0	0.545	0.685	11%
1	1.305	1.595	17%
2	2.095	2.525	22%
5	4.535	5.435	36%
10	8.375	8.995	63%
20	16.075	16.995	118%
30	23.835	24.995	180%
50	39.445	40.995	374%

Table 1: Liquidity Premium to Transaction Costs (LPTC) ratio changes dramatically as the level of fund manager's ambiguity aversion increases. Following parameters are used: risk-free rate $r = 0.10$, expected rate of the stock return $\mu = 0.15$, stock volatility $\sigma = 0.2$, risk aversion $\gamma = 2$, investment period $T = 10$, transaction costs $\alpha = \beta = 0.005$.

on trading behavior and liquidity premium with various levels of ambiguity aversion. LPTC ratio dramatically increases as ambiguity aversion mounts up. Under the base parameters, LPTC ratio even reaches 374%, which is larger than the empirical result of Amihud and Mendelson (1986), with a reasonable level of ambiguity aversion $\theta = 50$ proposed by Maenhout (2004). The second and third columns of Table 1 display, respectively, the lower and upper bounds of the no-transaction region.

Table 2 shows LPTC ratio with various sets of parameters. We use the baseline parameter values; interest rate $r = 0.10$, expected return $\mu = 0.15$, volatility $\sigma = 0.2$, risk aversion $\gamma = 2.0$, and transaction cost rates $\alpha = \beta = 0.005$. We perturb each parameter to test the sensitivity of LPTC ratio with respect to the parameter. Each row displays the value of LPTC ratio when a certain parameter is slightly changed from the base level. The sensitivities of LPTC ratio with respect to parameters are well visualized by Figures 6, 7, 8, and 9. Unfavorable investment opportunities yield a high LPTC ratio. Specifically, the lower the μ and α, β, the higher the LPTC. Also, the higher the r, γ and σ, the higher the LPTC. LPTC has a tendency to increase with respect to the ambiguity aversion regardless of changes in various parameters.

Figure 6 displays LPTC ratio against the level of ambiguity aversion θ for different levels

Parameters		the level of fund manager's ambiguity aversion θ					
		0	5	10	15	20	40
Base Parameter		10.9092	35.8948	62.7655	89.6349	117.7860	257.2725
$\mu = 0.15$	-0.02	14.0277	40.0462	66.8838	102.4801	130.4217	
	+0.02	6.8706	34.2433	61.0735	84.6838	109.0201	216.8429
$r = 0.10$	-0.02	6.8706	34.2433	61.0735	84.6838	109.0201	216.8429
	+0.02	14.0277	40.0462	66.8838	102.4801	130.4217	
$\gamma = 2.0$	-0.5	7.0225	33.7207	60.9612	83.6401	111.4706	254.0268
	+0.5	14.2086	38.0177	64.4669	91.4100	119.6901	260.3231
$\sigma = 0.2$	-0.5	6.1637	20.1063	32.5158	47.0311	63.4188	123.6786
	+0.5	19.5004	62.6824	99.3471	148.7677	205.2927	
$\alpha = \beta = 0.5\%$	-0.2	14.2156	56.9468	97.5115	140.2272	185.1502	405.7250
	+0.2	9.4609	27.6606	49.2933	69.2694	85.4915	195.3705

Table 2: **Comparative Statistics.** The base case parameter values are: risk-free rate $r = 0.10$, expected rate of the stock return $\mu = 0.15$, stock volatility $\sigma = 0.2$, risk aversion $\gamma = 2.0$, transaction cost rates $\alpha = \beta = 0.005$. Each row displays LPTC ratio when a certain parameter value is slightly changed. The second column presents the absolute size of perturbation.

of risk aversion γ. LPTC ratio rises as γ or θ increases. Liquidity premium is a kind of compensation given to fund managers when they trade a stock in the presence of transaction costs, so more risk or ambiguity-averse fund managers would require higher compensation to trade the stock in exchange for the market without transaction costs because they are not willing to participate in the stock market due to their higher risk or ambiguity aversion. Notice that a higher risk aversion induces fund managers to reduce their risky investment, which is observed in Figure 4 (A). Especially, when the ambiguity aversion is incorporated, fund managers are unwilling to participate in the stock market, which results in significantly declined risky investment (Figure 4 (B)). Hence, LPTC might sharply increase due to the higher ambiguity aversion of fund managers.

We can confirm that LPTC ratio does not change when the sum of fund manager's risk aversion γ and ambiguity aversion θ is constant. This means that risk aversion and ambiguity aversion has similar impact on LPTC ratio. For example, in Table 3, $\gamma = 1.5$, $\theta = 5.5$ and $\gamma = 2.0$, $\theta = 5.0$ yield very similar LPTC value of 35.8941 and 35.8948, respectively.

Figures 7, 8 and 9 plot LPTC ratio against the level of fund manager's ambiguity aversion with different levels of expected rate of stock return μ, stock volatility σ, and transaction costs

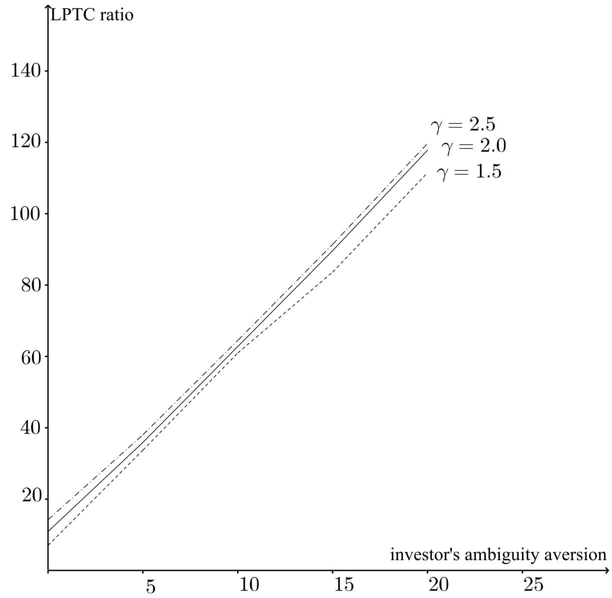


Figure 6: **The optimal trading strategy for the stock as a function of time** t: $r = 0.10$, $\mu = 0.15$, $\sigma = 0.2$, $\gamma = 2$, $T = 10$, $\alpha = 0.005$, $\beta = 0.005$ are used for parameter values.

γ	the level of fund manager's ambiguity aversion θ		
	4.5	5.0	5.5
1.5	31.5236	33.7207	35.8941
2.0	33.7226	35.8948	38.0477
2.5	35.8960	38.0177	40.1835

Table 3: LPTC ratio with different levels of fund manager's risk aversion γ and ambiguity aversion θ.

α and β, respectively.[9] The figures show that the impacts of perturbation in the parameter set $\{\mu, \sigma, \alpha = \beta\}$ dramatically change with different levels of ambiguity aversion θ. Especially, Figure 8 shows that, when the level of fund manager's ambiguity aversion θ is sufficiently large, 12.5% increase in the volatility from the baseline parameter, $\sigma = 0.20$, can generate almost 100% increase in LPTC ratio. The sensitivity of LPTC ratio subject to each parameter becomes much larger when we incorporate fund manager's ambiguity aversion into the model.

[9] We do not present a graph of LPTC ratio as a function of the level of ambiguity aversion θ with various levels of risk-free rate r because the result is same with Figure 7. This is because LPTC ratio depends on the risk premium $\mu - r$.

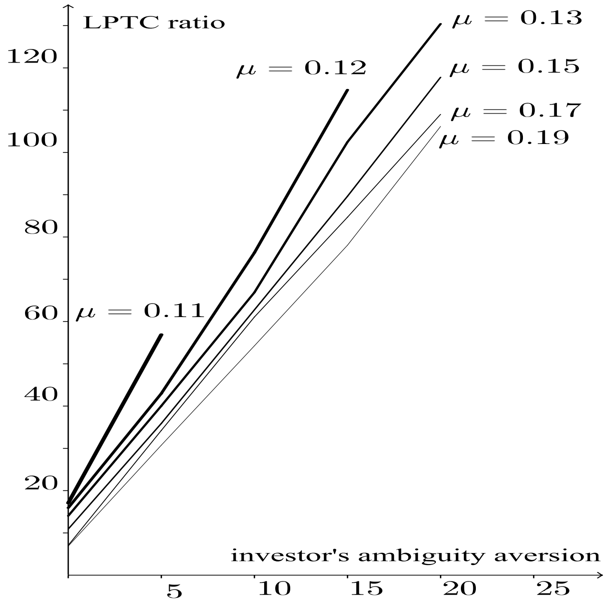


Figure 7: **The optimal trading strategy for the stock as a function of time** t: $r = 0.10$, $\mu = 0.15$, $\sigma = 0.2$, $\gamma = 2$, $T = 10$, $\alpha = 0.005$, $\beta = 0.005$ are used for parameter values.

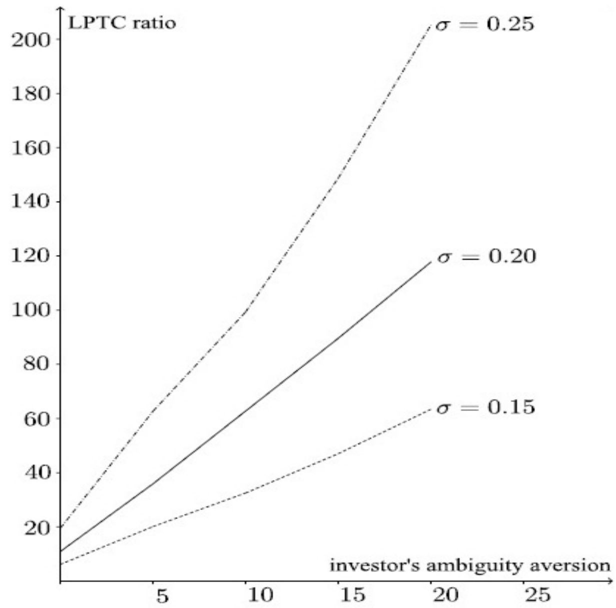


Figure 8: **The optimal trading strategy for the stock as a function of time** t: $r = 0.10$, $\mu = 0.15$, $\sigma = 0.2$, $\gamma = 2$, $T = 10$, $\alpha = 0.005$, $\beta = 0.005$ are used for parameter values.

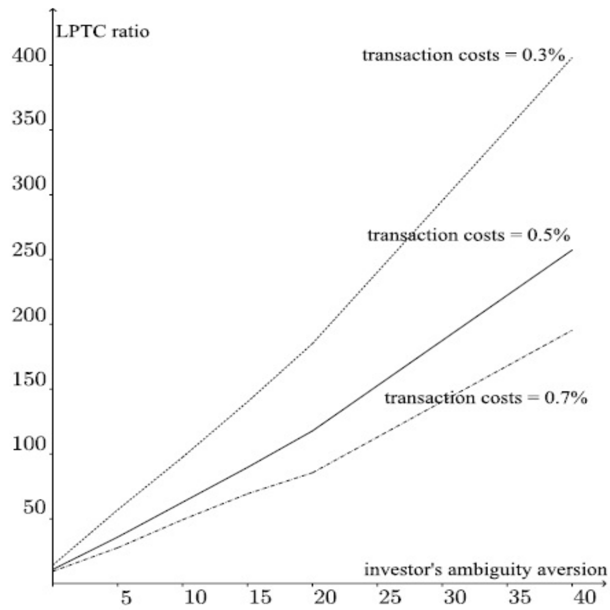


Figure 9: **The optimal trading strategy for the stock as a function of time** t: $r = 0.10$, $\mu = 0.15$, $\sigma = 0.2$, $\gamma = 2$, $T = 10$, $\alpha = 0.005$, $\beta = 0.005$ are used for parameter values.

4.4 Liquidity Crash in the Global Crisis and Ambiguity Aversion

The global crisis in 2008 has given a significantly negative impact on the financial economy, resulting in extreme events such as liquidity crash, credit crunch and flash crash. The common characteristic among these events that may happen during economic crises is global liquidity dry-up, resulting in banks, individuals, firms, capital market participants and financial institutions are becoming reluctant to conduct financial transactions. Indeed, loan securitization was rarely possible in the subprime mortgage crisis (Berger and Bouwman 2009). Acharya *et al.* (2011) and Perotti and Suarez (2011) demonstrate that exposure to liquidity risk in the financial crisis could result in the liquidity dry-up.

It is very important to explore what causes liquidity crash during economic crises. Many researchers have been trying to explain the liquidity crunch through the wealth effects stemming decreasing absolute risk aversion (Kyle and Xiong 2001), tighter financing constraints (Gromb and Vayanos 2002, Brunnemeier and Pedersen 2008), strategic intermediation across segmented markets (Rahi and Zigrand 2007), the asset holdings of banks consisting of mainly shorter maturity instruments, and adverse selection for dealers in the days preceding the flash crash (Easley *et al.* 2011). More recently, Cespa and Foucault (2012) illustrate the liquidity crash as a switch from the equilibrium with high liquidity to the equilibrium with low liquidity without any clear reason. Moreover, Malherbe (2013) argues the liquidity requirement suggested by Basel III might lead to liquidity dry-up because the requirement decreases the future need to gather cash.

The results in this section are compatible with their counterparts in Caballero and Krishnamurthy (2007), Routledge and Zin (2009),and Easley and O'Hara (2010) in that ambiguity aversion could have a significant impact on economic status, sometimes yielding liquidity crash. Especially, we are in keeping with observations of Routledge and Zin (2009) and Easley and O'Hara (2010) in which ambiguity-averse investors are not willing to trade in the stock market because of the increased bid-ask spread by the ambiguity. The higher LPTC generated in our model also supports the puzzling feature such as liquidity dry-up and flight to quality that there was little or no trading in risky assets. Ambiguity-averse fund managers would require a significant compensation to trade in exchange for no-transaction costs, so they are reluctant to trade in the stock market.

Specifically, our model can demonstrate the behaviors of myopic ambiguity-averse fund manager under different market situations. Table 4 shows LPTC ratio for various ambiguity aversion θ under two different market conditions; 'Down Markets' and 'Up Markets'. Following Ang and Bekaert (2002), we use risk aversion of $\gamma = 2$, risk-free rate of $r = 0.05$, expected rate of the stock return of $\mu = 0.1394$ and stock volatility of $\sigma = 0.2600$ at times of down markets, and $\mu = 0.1394$ and $\sigma = 0.1313$ at times of up markets[10].

In the table, myopic ambiguity-averse fund manager shows significantly different behaviors depending on the current market conditions. The manager requires greater liquidity premium during the down markets than during the up markets, even though ambiguity aversion θ does not change. For example, the manager with moderate ambiguity aversion, $\theta = 5$, would like to obtain 55.4609 LPTC in exchange for the market without transaction costs in the up markets. However, she needs 105.9377 LPTC in the down markets, which is about two times of that in the up markets.

A higher ambiguity aversion θ would be closely associated with a puzzling feature during

[10]The parameter values are those in Jang *et al.* (2007).

θ	Up Markets	Down Markets
0	74.5180	58.1376
5	55.4609	105.9377
10	66.8425	149.1123
20	87.7765	250.5166
30	108.3837	354.9056
40	132.3426	510.0148

Table 4: **LPTC ratio with different levels of ambiguity aversion θ under Up Markets and Down Markets**: $\gamma = 2$, $r = 0.05$, $\mu = 0.1394$ (0.1394), $\sigma = 0.1313$ (0.2600) are used for the parameter values under Up Markets (Down Markets, respectively).

the economic crisis that the financial market was very illiquid. The fund manager with higher ambiguity aversion requires greater liquidity premium at times of down markets than at times of up markets. For instance, the manager with $\theta = 40$ has LPTC ratio of 132.3426 during the up markets, but the ratio skyrockets to 510.0148 when markets turn down, which is about four times of that in the up markets. This can be interpreted as that the manager who are very cautious about her estimates of the first moment of stock return would be reluctant to trade in the stock market when it is bearish unless she obtains a significant compensation to trade in exchange for no-transaction costs. When an economic recession starts, ambiguity-averse fund manager's lack of trading could cause liquidity crash.

5 Conclusion

Traditional theories report that transaction costs only have a second-order effect on liquidity premium. Specifically, LPTC of Constantinides (1979) is about only 0.07, however market-data-based LPTC of Amihud and Mendelson (1986) is about 1.9. We find robust portfolio rules for ambiguity-aversive fund managers in a financial market with transaction costs. The model proposed in this paper permit liquidity premium much bigger than those found by most empirical literature. Using reasonably-calibrated parameters, we find liquidity premium obtained from the model is much bigger, so transaction costs can have a significant effect on investors' optimal investment behaviors.

The global crisis in 2008 has led to large bad effect on the financial economy, such as liquidity crash, credit crunch etc. Many researchers have been trying to explain for the liquidity crash in terms of various aspects. This paper provides an explanation for the liquidity crash through ambiguity aversion of fund managers. We assume fund managers are myopic because regarding fund performance they are evaluated for a year or 6-months. Using reasonably calibrated parameters, we find that the fund manager with a higher ambiguity aversion requires much bigger liquidity premium at times of down markets than at times of up markets. When an economic recession starts, the non-trading of ambiguity-averse fund manager incurred from the higher liquidity premium could cause liquidity crash or liquidity dry-up.

6 Appendix

6.1 Proof of Theorem 2.1

Applying HJB equation to (1) yields

$$0 = \sup_{\pi} \left(V_t^{RC} + (r + (\mu - r)\pi) w V_w^{RC} + \frac{1}{2}\sigma^2 \pi^2 w^2 V_{ww}^{RC} \right),$$

where $V_t^{RC} = \dfrac{\partial V^{RC}}{\partial t}$, $V_w^{RC} = \dfrac{\partial V^{RC}}{\partial w}$, $V_{ww}^{RC} = \dfrac{\partial^2 V^{RC}}{\partial w^2}$. When we consider a fund manager's ambiguity aversion, we can modify the above HJB equation with homothetic robustness as Maenhout (2004) did:

$$0 = \sup_{\pi} \inf_{h} \left(V_t^{RC} + (r + (\mu - r)\pi) w V_w^{RC} + \frac{1}{2}\sigma^2 \pi^2 w^2 V_{ww}^{RC} + V_w^{RC} \pi^2 \sigma^2 w^2 h + \frac{1}{2\Psi^{RC}} \sigma^2 \pi^2 w^2 h^2 \right),$$

where

$$\Psi^{RC}(x, y, t) = \frac{\theta}{(1-\gamma) V^{RC}(x, y, t)}.$$

From the first order conditions for h and π, respectively, we can obtain

$$h = -\Psi^{RC} V_w^{RC} \qquad \text{and} \qquad \pi = -\frac{V_w^{RC}}{(V_{ww}^{RC} - \Psi^{RC}(V_w^{RC})^2) w} \frac{\mu - r}{\sigma^2}. \tag{9}$$

Therefore, the value function of the fund manager with ambiguity aversion θ is the solution of the following partial differential equation:

$$0 = V_t^{RC} + rwV_w^{RC} - \frac{1}{2}\frac{{V_w^{RC}}^2}{V_{ww}^{RC} - \Psi^{RC}{V_w^{RC}}^2}\Big(\frac{\mu - r}{\sigma}\Big)^2.$$

Guess value function $V^{RC}(t, w)$ as

$$V^{RC}(t, w) = f(t)\frac{w^{1-\gamma}}{1-\gamma}. \tag{10}$$

Then $f(t)$ is completely determined by

$$f(t) = e^{a_0(T-t)}, \qquad \text{where } a_0 = (1-\gamma)\Big(r + \frac{1}{2}\frac{1}{\gamma+\theta}\Big(\frac{\mu - r}{\sigma}\Big)^2\Big).$$

Substituting $V^{RC}(t, w)$ in (10) into optimal investment π in (9) gives

$$\pi = \frac{1}{(\gamma+\theta)}\frac{\mu - r}{\sigma^2}.$$

Q.E.D.

6.2 Proof of Theorem 3.1

Define

$$v(z, t) \equiv \psi_z(z, t).$$

Then $v(z, t)$ satisfies the following parabolic double obstacle problem:

$$\begin{aligned} -v_t - \mathcal{L}v = 0,\ \text{if}\ \ & \frac{1}{z+1+\alpha} < v < \frac{1}{z+1-\beta}, \\ -v_t - \mathcal{L}v \geq 0,\ \text{if}\ \ & v = \frac{1}{z+1+\alpha}, \\ -v_t - \mathcal{L}v \leq 0,\ \text{if}\ \ & v = \frac{1}{z+1-\beta}, \\ v(z, T) = & \frac{1}{z+1-\beta},\quad z \in (\beta - 1, +\infty), \end{aligned} \tag{11}$$

where

$$\mathcal{L}v = \frac{1}{2}\sigma^2 z^2 v_{zz} - \big((\mu-r) - \sigma^2(\gamma+\theta+1)\big)zv_z - (\mu - r - \sigma^2(\gamma+\theta))v + \sigma^2(1-\gamma-\theta)zv^2 + \sigma^2(1-\gamma-\theta)z^2 v v_z.$$

According to Dai and Yi (2009), we can obtain the following inequalities:

$$v_t \geq 0, \tag{12}$$

$$v_z + v^2 \leq 0. \tag{13}$$

Using the inequality in (13) gives

$$\frac{\partial}{\partial z}\Big(v - \frac{1}{z+1+\alpha}\Big) = v_z + \frac{1}{(z+1+\alpha)^2} \leq v_z + v^2 \leq 0.$$

If $(z_1, t) \in \mathbf{BR}$, then

$$0 = v(z_1, t) - \frac{1}{z_1+1+\alpha} \geq v(z,t) - \frac{1}{z+1+\alpha} \geq 0,$$

for any $z \geq z_1$. Thus, $v(z,t) - \dfrac{1}{z+1+\alpha} = 0$ and $(z,t) \in \mathbf{BR}$. This proves (5). Also, the inequality in (13) implies

$$\frac{\partial}{\partial z}\Big[(z+1-\beta)^2\Big(v - \frac{1}{z+1-\beta}\Big)\Big] = (z+1-\beta)^2(v_z+v^2) - \{(z+1-\beta)v-1\}^2 \leq 0.$$

Then we can prove (4) similarly. Utilizing the inequality in (12) yields that if $(z, t_1) \in \mathbf{BR}$, then

$$0 = v(z, t_1) - \frac{1}{z+1+\alpha} \geq v(z,t) - \frac{1}{z+1+\alpha} \geq 0,$$

for any $t \leq t_1$. Then we can obtain $v(z,t) - \dfrac{1}{z+1+\alpha} = 0$, and, thus, $(z,t) \in \mathbf{BR}$. This proves the monotonic property of $z_b(t)$. Similarly, we can prove the monotonicity of $z_s(t)$. Recall that

$$-v_t - \mathcal{L}v \geq 0, \quad \text{if} \quad v = \frac{1}{z+1+\alpha}.$$

Then for any $(z,t) \in \mathbf{BR}$, i.e., $v(z,t) = \dfrac{1}{z+1+\alpha}$, we can get

$$\begin{aligned} 0 &\leq -\frac{\partial}{\partial t}\Big(\frac{1}{z+1+\alpha}\Big) - \mathcal{L}\Big(\frac{1}{z+1+\alpha}\Big) \\ &= -\mathcal{L}\Big(\frac{1}{z+1+\alpha}\Big) = \frac{(1+\alpha)}{(z+1+\alpha)^3}[(\mu-r)z + (\mu - r - \sigma^2(\gamma+\theta))(1+\alpha)]. \end{aligned}$$

Therefore,

$$z \geq -\frac{\mu - r - \sigma^2(\gamma+\theta)}{\mu - r}(1+\alpha) = (1+\alpha)z_M^\theta.$$

Similarly, we can prove

$$z \leq -\frac{\mu - r - \sigma^2(\gamma+\theta)}{\mu - r}(1-\beta) = (1-\beta)z_M^\theta,$$

for any $(z,t) \in \mathbf{SR}$. To show $z_s(T^-) = (1-\beta)z_M^\theta$, suppose contrary, i.e., $z_s(T^-) < (1-\beta)z_M^\theta$. For any $z \in (z_s(T^-), (1-\beta)z_M^\theta)$, we can get

$$\begin{aligned}
\left.\frac{\partial v}{\partial t}\right|_{t=T} &= -\mathcal{L}v|_{t=T} \\
&= \frac{(1-\beta)}{(z+1-\beta)^3}[(\mu-r)z + (\mu - r - \sigma^2(\gamma+\theta))(1-\beta)] < 0.
\end{aligned}$$

This contradicts to the inequality in (12). Hence, $z_s(T^-) = (1-\beta)z_M^\theta$. It remains to prove (8). Define

$$y \equiv \frac{z}{z+1+\alpha}, \quad \tilde{v}(y,t) \equiv \Big(v(z,t) - \frac{1}{z+1+\alpha}\Big)\frac{(z+1+\alpha)^2}{1+\alpha}.$$

Then the parabolic double obstacle problem in (11) can be written as

$$\begin{aligned}
&-\tilde{v}_t - \mathcal{L}_y\tilde{v} = 0, \text{ if } 0 < \tilde{v} < \frac{\alpha+\beta}{(1-\beta)+(\alpha+\beta)y}, \\
&-\tilde{v}_t - \mathcal{L}_y\tilde{v} \geq 0, \text{ if } \tilde{v} = 0, \\
&-\tilde{v}_t - \mathcal{L}_y\tilde{v} \leq 0, \text{ if } \tilde{v} = \frac{\alpha+\beta}{(1-\beta)+(\alpha+\beta)y}, \\
&\qquad \tilde{v}(z,T) = \frac{\alpha+\beta}{(1-\beta)+(\alpha+\beta)y}, \quad y \in (\frac{\beta-1}{\alpha+\beta}, 1),
\end{aligned} \tag{14}$$

where

$$\begin{aligned}
\mathcal{L}_y\tilde{v} = &\frac{1}{2}\sigma^2y^2(1-y)^2\tilde{v}_{yy} - \{\mu - r - \sigma^2(\gamma+\theta+1) + 3\sigma^2 y - \sigma^2(1-\gamma-\theta)\}y(1-y)\tilde{v}_y \\
&- \{\mu - r - \sigma^2(\gamma+\theta) - 2(\mu - r - 2\sigma^2(\gamma+\theta))y + 3\sigma^2(\gamma+\theta)y^2\}\tilde{v} \\
&+ \{\sigma^2(1-\gamma-\theta) - 2\sigma^2(1-\gamma-\theta)y\}y(1-y)\tilde{v}^2 + \sigma^2(1-\gamma-\theta)y^2(1-y)^2\tilde{v}\tilde{v}_y \\
&- \{\mu - r - \sigma^2(\gamma+\theta) - (\mu - r - 2\sigma^2(\gamma+\theta))y - \sigma^2(\gamma+\theta)y^2\}\frac{1}{(1-y)}.
\end{aligned}$$

Let $C(t) \equiv \tilde{v}(1,t)$. The upper bound of sell boundary $z_s(t)$ in (6) implies $C(t) < \frac{\alpha+\beta}{1+\alpha}$ for all t. The parabolic double obstacle problem in (14) reduces to

$$\begin{aligned}
&-C'(t) - (\mu-r)C(t) + (\mu-r) = 0, \text{ if } 0 < C(t) < \frac{\alpha+\beta}{1+\alpha}, \\
&-C'(t) - (\mu-r)C(t) + (\mu-r) \geq 0, \text{ if } C(t) = 0, \\
&\quad C(T) = \frac{\alpha+\beta}{1+\alpha}.
\end{aligned}$$

Solving it, we can obtain

$$C(t) = \begin{cases} 1 - \dfrac{1-\beta}{1+\alpha} e^{(\mu - r)(T-t)}, & \text{when } \ t \geq t_b, \\ 0, & \text{when } \ t < t_b, \end{cases}$$

where $t_b = T - \dfrac{1}{(\mu - r)} \ln\left(\dfrac{1+\alpha}{1-\beta}\right)$. Therefore, $z_b(t) = +\infty$ for $t \in [t_b, T)$. **Q.E.D.**

6.3 Proof of Theorem 3.2

When $z = 0$, the parabolic double obstacle problem in (11) reduces to

$$\begin{aligned} &-v_t(0,t) + \delta_\theta v(0,t) = 0, \ \text{ if } \ \frac{1}{1+\alpha} < v(0,t) < \frac{1}{1-\beta}, \\ &-v_t(0,t) + \delta_\theta v(0,t) \geq 0, \ \text{ if } \ v = \frac{1}{1+\alpha}, \\ &-v_t(0,t) + \delta_\theta v(0,t) \leq 0, \ \text{ if } \ v = \frac{1}{1-\beta}, \\ &\ v(0,T) = \frac{1}{1-\beta}. \end{aligned}$$

Solving it, for $\delta_\theta > 0$ we can get

$$v(0,t) = \begin{cases} \dfrac{1}{1-\beta} e^{\delta_\theta (T-t)}, & \text{when } \ t > \hat{t}_b, \\ \dfrac{1}{1+\alpha}, & \text{when } \ t \leq \hat{t}_b, \end{cases}$$

where $\hat{t}_b = T - \dfrac{1}{\delta_\theta} \ln\left(\dfrac{1-\beta}{1+\alpha}\right)$. Thus, if $\delta_\theta > 0$, then $z_s(t) < 0$ for all t, $z_b(t) < 0$ for $t < \hat{t}_b$ and $z_b(t) > 0$ for $t \geq \hat{t}_b$. If $\delta_\theta \leq 0$, then $v(0,t) = \dfrac{1}{1-\beta}$. Therefore, if $\delta_\theta \leq 0$, then $z_s(t) > 0$ for all t. **Q.E.D.**

References

Acharya, V., A. Krishnamurthy, and E. Perotti. 2011. A Concensus View on Liquidity Risk. VOX, September 14, Available at http://www.voxeu.eu/article/consensus-view-liquidity-risk.

Amihud, Y., and H. Mendelson. 1986. Asset Pricing and the Bid-Ask Spread. *Journal of Financial Economics.* **17** 223–249.

Berger, A. N., and C. H. S. Bouwman. 2010. Bank Liquidity Creation, Monetary Policy, and

Financial Crises. Working Paper.

Brunnermeier, M. K. 2009. Deciphering the Liquidity and Credit Crunch 2007-2008. *Journal of Economic Perspectives.* **23** 77–100.

Brunnermeier, M. K., and L. H. Pedersen. 2009. Market Liquidity and Funding Liquidity. *Review of Financial Studies.* **22** 2202–2236.

Caballero, R. J., and Krishnamurthy, A. 2008. Collective Risk Management in a Flight to Quality Episode. *Journal of Finance.* **63** 2195–2230.

Cespa, G., and T. Foucault. 2012. Illiquidity Contagion and Liquidity Crashes. Working Paper.

Constantinides, G. M. 1979. Multiperiod Consumption and Investment Behavior with Convex Transaction Costs. *Management Science.* **25** 1127–1137.

Constantinides, G. M. 1986. Capital Market Equilibrium with Transaction Costs. *Journal of Political Economy.* **94** 842–862.

Dai, M., and F. Yi. 2009. Finite-Horizon Optimal Investment With Transaction Costs: A Parabolic Double Obstacle Problem. *Journal of Differential Equations.* **246** 1445–1469.

Dai, M., and Y. Zhong. 2010. Penalty Methods for Continuous-Time Portfolio Selection with Proportional Transaction Costs. *Journal of Computational Finance.* **13** 1–31.

Dai, M., P. Li, H. Liu, and Y. Wang. 2012. Market Closure and the Liquidity Premium Puzzle. Working Paper.

Davis, M. H. A., and A. R. Norman. 1990. Portfolio Selection with Transaction Costs. *Mathematics of Operations Research.* **15** 676–713.

Easley, D., P. M. López, and M. O'Hara. 2010. Liquidity and Valuation in an Uncertain World. *Journal of Financial Economics.* **97** 1–11.

Easley, D., P. M. López, and M. O'Hara. 2011. The Microstructure of the 'Flash Crash': Flow Toxicity, Liquidity Crashes, and the Probability of Informed Trading. *Journal of Portfolio*

Management. **37** 118–128.

Edmans, A. 2009. Blockholder Trading, Market Efficiency, and Managerial Myopia. *Journal of Finance.* **64** 2481–2513.

Ellsberg, D. 1961. Risk, Ambiguity and the Savage Axioms. *Quarterly Journal of Economics.* **75** 643–669.

Epstein, L. G., and M. Schneider. 2003. Recursive Multiple-Priors. *Journal of Economic Theory.* **113** 1–31.

Garlappi, L., R. Uppal, and T. Wang. 2007. Portfolio Selection with Parameter and Model Uncertainty: a Multi-Prior Approach. *Review of Financial Studies.* **20** 41–81.

Gilboa, I., and D. Schmeidler. 1989. Maxmin Expected Utility with Non-Unique Prior. *Journal of Mathematical Economics.* **18** 141–153.

Gompers, P. A. 1994. The Rise and Fall of Venture Capital. *Business and Economic History.* **23**

Gromb, D., and D. Vayanos. 2002. Equilibrium and Welfare in Markets with Financially Constrained Arbitrageurs. *Journal of Financial Economics.* **66** 361–407.

Hansen, L. P., and T. J. Sargent. 2001. Robust Control and Model Uncertainty. *American Economic Review.* **91** 60–66.

Hansen, L. P., and T. J. Sargent. 2011. Robustness and Ambiguity in Continuous Time. *Journal of Economic Theory.* **146** 1195–1223.

Hansen, L. P., T. J. Sargent, G. Turmuhambetova, and N. Williams. 2006. Robust Control and Model Misspecification. *Journal of Economic Theory.* **128** 45-90.

Jang, B. G., H. K. Koo, H. Liu, and M. Loewenstein. 2007. Liquidity Premia and Transaction Costs. *Journal of Finance.* **62** 2329–2366.

Ju, N., and J. Miao. 2012. Ambiguity, Learning, and Returns. *Econometrica.* **80** 559–591.

Kahneman, D., and A. Tversky. 1979. Prospect Theory: An Analysis of Decision under Risk.

Econometrica. **47** 263–292.

Keynes, M. 1936. The General Theory of Employment, Interest, and Money. *London Macmillan.*

Kinight, F. 1921. Risk, Ambiguity, and Profit. *Boston: Houghton Mifflin.*

Klibanoff, P., M. Marinacci, and S. Mukerji. 2005. A Smooth Model of Decision Making under Ambiguity. *Econometrica.* **73** 1849–1892.

Klibanoff, P., M. Marinacci, and S. Mukerji. 2011. Definitions of Ambiguous Events and the Smooth Ambiguity Model. *Economic Theory.* **48** 399–424.

Kyle, A., and W. Xiong. 2001. Contagion as a Wealth Effect. *Journal of Finance.* **56** 1401–1440.

Liu, J., J. Pan, and T. Wang. 2005. An Equilibrium Model of Rare-Event Premia and Its Implication for Option Smirks. *Review of Financial Studies.* **18** 131–164.

Liu, H., and M. Loewenstein. 2002. Optimal Portfolio Selection with Transaction Costs and Finite Horizons. *Review of Financial Studies.* **15** 805–835.

Lynch, A., and S. Tan. 2011. Explaining the Magnitude of Liquidity Premia: The Roles of Return Predictability, Wealth Shocks and State-dependent Transaction Costs. *Journal of Finance.* **66** 1329–1368.

Maenhout, P. J. 2004. Robust Portfolio Rules and Asset Pricing. *Review of Financial Studies.* **17** 951–983.

Malherbe, F. Self-fulfilling Liquidity Dry-Ups. *Journal of Finance forthcoming.*

Marston, C. L., and B. M. Craven. A Survey of Corporate Perceptions of Short-Termism among Analysts and Fund Managers. *European Journal of Finance.* **4**

Mehra, R., and E. C. Prescott. 1985. The Equity Premium: a Puzzle. *Journal of Monetary Economics.* **15** 145–161.

Perotti, E., and J. Suarez. 2011. A Pigovian Approach to Liquidity Regulation. *International*

Journal of Central Banking. **7** 3–41.

Rahi, R., and J. P. Zigrand. 2007. A Theory of Strategic Intermediation and Endogenous Liquidity. Working Paper.

Routledge, B. R., and S. E. Zin. 2009. Model Uncertainty and Liquidity. *Review of Economic Dynamics.* **12** 543–566.

Shiller, R. J. 1981. Do Stock Prices Move Too Much To Be Justified by Subsequent Changes in Dividends" *American Economic Review.* **71** 421–436.

Shreve, S. E., and H. M. Soner. 1994. Optimal Investment and Consumption with Transaction Costs. *The Annals of Applied Probability.* **4** 609–692.

Tversky, A., and D. Kahneman. 1992. Advances in Prospect Theory: Cumulative Representation of Uncertainty. *Journal of Risk and Uncertainty.* **5** 297–323.

Uppal, R., and T. Wang. 2003. Model Misspecification and Underdiversification. *Journal of Finance.* **58** 2465–2486.

Weil, P. 1989. The Equity Premium Puzzle and the Risk-Free Rate Puzzle. *Journal of Monetary Economics.* **24** 401–421.

Zhu, W. 2011. Ambiguity Aversion and an Intertemporal Equilibrium Model of Catastrophe-Linked Securities Pricing. *Insurance: Mathematics and Economics.* **49** 38–46.